Physics 564 - HOMEWORK SET #4

Solutions

1. (a) We can use the relativistic relation between force and acceleration \( F = m\gamma^3 a \), so for \( F = -bv \) we get \( \frac{dv}{dt} = a = -bv/(m\gamma^3) \) or by integrating

\[
\int_{v_0}^{v} \frac{dv}{v (1 - v^2/c^2)^{3/2}} = -(b/m) \int_0^t dt.
\]

The result is

\[
\frac{b}{m} t = \gamma_0 - \gamma + \frac{1}{2} \ln \left( \frac{\gamma + 1 + \gamma_0 - 1}{\gamma - 1 + \gamma_0 + 1} \right)
\]

where \( \gamma_0 = 1/\sqrt{1 - v_0^2/c^2} \). As \( v \to 0, \gamma \to 1 \) and \( t \to \infty \).

(b) The solution for the nonrelativistic case is easily found to be \( v = v_0 e^{-bt/m} \). Plotting the relativistic and non-relativistic solutions on a semi-log graph gives

![Relativistic vs Non-Relativistic Solutions](image)

We see that for high velocities the relativistic case is much different than the non-relativistic case, but once \( \beta \) is less than around 0.1, the relativistic case becomes approximately a straight line, the same as the non-relativistic case. The reason the velocity does not slow as fast initially as in the non–relativistic case is that the effective mass in the force equation, \( \gamma^3 m \), is bigger than the rest mass, and therefore the particle has more inertia and less deceleration. As \( v \to 0 \) and \( \gamma \to 1 \), the inertial mass approaches its non-relativistic limit.

2. (a) The energy function is \( h = \gamma mc^2 + V = mc^2 \sqrt{1 - v^2/c^2} + V \). Since \( V \) doesn’t explicitly depend on \( t \), \( h \) is conserved. Letting \( h = E \), a constant, then we can solve for \( v \) to get

\[
\frac{dx}{dt} = v = c \sqrt{1 - \frac{m^2 c^4}{(E-V)^2}}
\]

which can be rearranged and integrated to yield

\[
\int \frac{(E-V)dx}{\sqrt{(E-V)^2 - m^2 c^4}} = \int c \ dt.
\]

The turning points occur when \( v = 0 \) (and therefore \( \gamma = 1 \)), so if \( x = b \) is a turning point then \( E = mc^2 + Ab \). Then integrating \( x \) from 0 to \( b \) should give a quarter of a period, and

\[
\int_0^b \frac{(mc^2 + Ab - Ax)dx}{\sqrt{(mc^2 + Ab - Ax)^2 - m^2 c^4}} = \frac{1}{4} cT;
\]
where we have used the fact that \( |x| = x \) for \( 0 \leq x \leq b \). This may be integrated to give

\[
T = 4 \sqrt{2mb \over A} + {b^2 \over c^2}.
\]

(b) In the limit \( c \to \infty \) we can expand \( T \) in powers of \( c^{-2} \)

\[
T = 4 \sqrt{2mb \over A} \left(1 + \frac{Ab}{2mc^2}\right)^{1/2} \simeq 4 \sqrt{2mb \over A} \left(1 + \frac{Ab}{4mc^2}\right).
\]

Thus the leading (non-relativistic) term is \( T = 4 \sqrt{2mb/A} \). The Newtonian treatment (again considering the region \( 0 \leq x \leq b \)) has \( x = x_0 + at^2/2 \), where \( a \) is the acceleration. The value of \( x_0 \) is determined by energy conservation, \( mvt^2/2 = V(b) = Ab \), which implies \( x_0 = \sqrt{2Ab/m} \). The period is 4 times the time it takes to go from 0 to \( b \), which is found by setting \( x = b \) and solving for \( t \). The final result is \( T = 4 \sqrt{-2b/a} \).

The force is \( F = -dV/dx = -A \), so \( a = F/m = -A/m \) and \( T = 4 \sqrt{2mb/A} \), which agrees with the leading term in the relativistic expression. The first order fractional correction is \( Ab/4mc^2 \).

3. (a) First we find the energy and momentum of the proton (\( p \)) and pion (\( \pi \)) in the \( \Lambda \) rest frame (i.e., CM frame)

\[
\begin{align*}
E_p &= (m_{\Lambda}^2 + m_p^2 - m_\pi^2)/(2m_{\Lambda}) = 943.4 \text{ MeV} \\
p_p &= \sqrt{E_p^2 - m_p^2} = 100.9 \text{ MeV} \\
E_\pi &= m_\Lambda - E_p = 172.6 \text{ MeV} \\
p_\pi &= \sqrt{E_\pi^2 - m_\pi^2} = 100.9 \text{ MeV}
\end{align*}
\]

where we are letting \( c = 1 \) for simplicity. Note that \( p_p = p_\pi \) as it should in the CM frame. For the proton we transform its 4-momentum into the lab frame (I only consider motion in the \( x-z \) plane)

\[
\begin{pmatrix}
E'_p \\
p_{xp}' \\
p_{yp}' \\
p_{zp}'
\end{pmatrix} = \begin{pmatrix}
\gamma & 0 & 0 & \gamma \beta \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\gamma \beta & 0 & 0 & \gamma
\end{pmatrix} \begin{pmatrix}
E_p \\
p_p \sin \theta \\
0 \\
p_p \cos \theta
\end{pmatrix} = \begin{pmatrix}
\gamma (E_p + \beta p_p \cos \theta) \\
p_p \sin \theta \\
0 \\
\gamma (\beta E_p + p_p \cos \theta)
\end{pmatrix}
\]

where \( \gamma = E_{\Lambda}/m_{\Lambda} = 1.1649 \) and \( \beta = \sqrt{1 - 1/\gamma^2} = 0.5129 \) are the Lorentz factors for the \( \Lambda \) particle. A positive \( z \) component then requires \( \beta E_p + p_p \cos \theta > 0 \) or \( \cos \theta > -\beta E_p/p_p = -4.80 \), which is always true. Therefore the proton will always be in the forward hemisphere (even if it is in the \( -z \) direction in the CM frame!). For the pion we get a similar formula, except that the pion momentum is back-to-back with the proton, which means that \( \theta_\pi = \pi - \theta \), or \( \beta E_\pi + p_\pi \cos(\pi - \theta) > 0 \), which leads to \( \cos \theta < \beta E_\pi/p_\pi = 0.877 \), or \( \theta > 29^\circ \). Thus if the proton has \( \theta > 29^\circ \), then both particles will be in the forward hemisphere.

(b) The energy of the pion in the rest frame is \( E'_\pi = \gamma (E_\pi - \beta p_\pi \cos \theta) \), where \( \theta \) is the angle the proton makes with respect to the \( z \) axis (see above). Thus the Lorentz factor of the pion in the CM frame is

\[
\gamma_\pi = E'_\pi/m_\pi = \gamma (E_\pi - \beta p_\pi \cos \theta)/m_\pi.
\]

Taking the range \(-1 \leq \cos \theta \leq 1 \) we find that \( 1.0055 \leq \gamma_\pi \leq 1.867 \). Since the lifetime of the pion in the lab frame is \( \tau_\pi \) times the lifetime in its rest frame, the observed lifetimes must lie in the range \( 2.61 \times 10^{-8} \text{ s} \leq \tau_\pi \leq 4.85 \times 10^{-8} \text{ s} \).

4. For the process \( A + B \to C + D \), let the initial four-momenta be \( p_A = (E_A, \textbf{p}_A) \) and \( p_B = (m, 0) \) (since B is initially at rest) and the final four-momenta be \( p_C = (E_C, \textbf{p}_C) \) and \( p_D = (E_D, \textbf{p}_D) \) (space momenta are indicated by boldface). For simplicity I have set \( c = 1 \). To determine if \( \textbf{p}_C \) and \( \textbf{p}_D \) are
perpendicular, we calculate $\mathbf{p}_C \cdot \mathbf{p}_D$. Conservation of the time-like component of four-momentum gives $E_D = E_A + m - E_C$ and $\mathbf{p}_D = \mathbf{p}_A - \mathbf{p}_C$. The invariant mass relation for particle D gives

\begin{align*}
m^2 &= p_D^2 = (E_D)^2 - |\mathbf{p}_D|^2 \\
&= (E_A/c + m - E_C)^2 - (\mathbf{p}_A - \mathbf{p}_C)^2 \\
&= E_A^2 + m^2 + E_C^2 + 2mE_A - 2mE_C - 2E_AE_C - |\mathbf{p}_A|^2 - |\mathbf{p}_C|^2 + 2\mathbf{p}_A \cdot \mathbf{p}_C.
\end{align*}

Using the relations $E_A^2 - |\mathbf{p}_A|^2 = m^2$, etc.,

$$m^2 = 3m^2 + 2mE_A - 2mE_C - 2E_AE_C + 2\mathbf{p}_A \cdot \mathbf{p}_C$$

or

$$\mathbf{p}_C \cdot \mathbf{p}_D = E_AE_C + m(E_C - E_A) - m.$$

Therefore

\begin{align*}
\mathbf{p}_C \cdot \mathbf{p}_D &= \mathbf{p}_C \cdot (\mathbf{p}_A - \mathbf{p}_C) = \mathbf{p}_A \cdot \mathbf{p}_C - |\mathbf{p}_C|^2 \\
&= E_AE_C + m(E_C - E_A) - m - |\mathbf{p}_C|^2 \\
&= E_AE_C + m(E_C - E_A) - E_C^2 \\
&= (E_A - E_C)(E_C - m).
\end{align*}

Thus the only way $\mathbf{p}_C \cdot \mathbf{p}_D = 0$ is for $E_C = E_A$ (the moving particle does not exchange any energy with the other one, which can be interpreted as there being no collision, so the particle at rest remains at rest) or $E_C = m$ (the moving particle stops, which means that the other particle proceeds with the original velocity of the first). In both cases, the dot product of the space momenta is zero because one of the particles is at rest in the final state, not because the final state momenta are perpendicular. Therefore, the outgoing momenta cannot be perpendicular.