Physics 564 - HOMEWORK SET #12
Solutions

1. (a) The Hamiltonian is $H = \frac{p^2}{(2m)} + mgy$, with unperturbed part $H_0 = \frac{p^2}{(2m)}$ and perturbation $\Delta H = mgy$. Since $H$ is independent of time, we take the generating function as $S(y,t) = W(y) - \alpha^2 t/(2m)$, where $\alpha^2/(2m)$ is the constant value of the unperturbed Hamiltonian. For $H_0$ the particle is free and $P$ is a constant, so the generating function of the time evolution is just the identity, i.e., $W(y) = \alpha y$ (remembering $\alpha$ is the constant value of the momentum). Therefore $S = \alpha y - \alpha^2 t/(2m)$.

The new coordinate has constant value (say $\beta$) and is given by $\beta = Q = \frac{\partial S}{\partial \alpha} = y - \alpha t/m$, so $y = \beta + \alpha t/m$ and so $\Delta H = mgy = mg(\beta + \alpha t/m)$. The first order time-dependent perturbation theory equations are

$$\dot{\alpha}_1 = -\left(\frac{\partial \Delta H}{\partial \beta}\right)_{\alpha_0,\beta_0} = -mg$$

$$\dot{\beta}_1 = -\left(\frac{\partial \Delta H}{\partial \alpha}\right)_{\alpha_0,\beta_0} = gt.$$  

Integrating these to find $\alpha_1$ and $\beta_1$

$$\alpha_1 = \alpha_0 - mgt, \quad \beta_1 = \beta_0 + \frac{1}{2}gt^2.$$  

Therefore

$$x_1 = \beta_1 + \alpha_1 t/m = \beta_0 + \frac{\alpha_0}{m}t - \frac{1}{2}gt^2$$

and

$$p_1 = \alpha_1 = \alpha_0 - mgt.$$  

(b) At second order

$$\dot{\alpha}_2 = -\left(\frac{\partial \Delta H}{\partial \beta}\right)_{\alpha_1,\beta_1} = -mg$$

$$\dot{\beta}_2 = -\left(\frac{\partial \Delta H}{\partial \alpha}\right)_{\alpha_1,\beta_1} = gt.$$  

Since $\dot{\alpha}_2$ and $\dot{\beta}_2$ are the same as $\dot{\alpha}_1$ and $\dot{\beta}_1$, respectively, then $\alpha_2 = \alpha_1$ and $\beta_2 = \beta_1$. Similarly $\alpha_n = -mg$ and $\beta_n = gt$, so the $n$th order solutions will also be the same, i.e., $\alpha_n = \alpha_1$ and $\beta_n = \beta_1$. This makes sense since the first-order solution is the exact answer in this case.

2. (a) The Hamiltonian may be written

$$H = \frac{p^2}{2mL^2} + \frac{mgL}{2} \theta^2 - \frac{mgL}{24} \theta^4 + \frac{mgL}{720} \theta^6.$$  

To put this in the usual form for time-independent perturbation theory $H = H_0 + \epsilon H_1 + \epsilon^2 H_2$, we let $\epsilon = \theta_1^2$, where $\theta_1$ is the maximum amplitude in the unperturbed problem. Then

$$H_0 = \frac{p^2}{2mL^2} + \frac{mgL}{2} \theta^2, \quad H_1 = -\frac{mgL}{24} \theta^4, \quad H_2 = \frac{mgL}{720} \theta^6.$$
In action angle variables \((w_0, J_0)\) for \(H_0\), we have solutions

\[
\theta = \sqrt{\frac{J_0}{\pi mL^2 \omega_0}} \sin 2\pi w_0, \quad p_\theta = \sqrt{\frac{mL^2 \omega_0}{\pi}} \cos 2\pi w_0
\]

where \(\omega_0 = \sqrt{g/L}\) or \(g = L\omega_0^2\). Note also that \(\theta_i^2 = J_0/(\pi mL^2 \omega_0)\) from the amplitude of \(\theta\). In terms of \(w_0\) and \(J_0\), we have \(H_0 = \nu_0 J_0 = \omega_0 J_0/(2\pi)\) and

\[
H_1 = -\frac{mL^2 \omega_0^2}{24} \frac{1}{\theta_1^4} \left( \frac{J_0}{\pi mL^2 \omega_0} \right)^2 \sin^4 2\pi w_0, \quad H_2 = \frac{mL^2 \omega_0^2}{720} \frac{1}{\theta_1^6} \left( \frac{J_0}{\pi mL^2 \omega_0} \right)^3 \sin^6 2\pi w_0.
\]

The general Hamiltonian is \(\alpha = \alpha_0 + \epsilon \alpha_1 + \epsilon^2 \alpha_2\), where \(\alpha_0 = H_0\) and

\[
\alpha_1 = \overline{H_1}, \quad \alpha_2 = \overline{H_2} + \frac{\partial Y_1}{\partial w_0} \frac{\partial H_1}{\partial J_0} + \frac{1}{2} \left( \frac{\partial Y_1}{\partial w_0} \right)^2 \frac{\partial^2 H_1}{\partial J_0^2}
\]

and the bar indicates a cycle average over the unperturbed cycle. Also

\[
\nu_0 \frac{\partial Y_1}{\partial w_0} = \overline{H_1} - H_1 \implies \frac{\partial Y_1}{\partial w_0} = \frac{2\pi}{\omega_0} \overline{(H_1 - H_1)}.
\]

Note \(H_0\) is linear in \(J_0\), so \(\partial^2 H_0/\partial J_0^2 = 0\). Then

\[
\alpha_1 = -\frac{mL^2 \omega_0^2}{24} \frac{1}{\theta_1^4} \left( \frac{J_0}{\pi mL^2 \omega_0} \right)^2 \sin^4 2\pi w_0 = -\frac{J_0^2}{64\pi^2 mL^2 \theta_1^2}
\]

where we have used \(\overline{\sin^4 \phi} = 3/8\). Next we have

\[
\alpha_2 = \frac{J_0^3}{720\pi^3 m^2 L^4 \theta_1^4 \omega_0} \overline{\sin^6 2\pi w_0} + \frac{2\pi}{\omega_0} \overline{(H_1 - H_1)} \frac{\partial H_1}{\partial J_0}
\]

\[
\alpha_2 = \frac{J_0^3}{720\pi^3 m^2 L^4 \theta_1^4 \omega_0} \overline{\sin^6 2\pi w_0} + \frac{2\pi}{\omega_0} \overline{H_1} \frac{\partial H_1}{\partial J_0} - \frac{2\pi}{\omega_0} \frac{1}{2} \frac{\partial^2 H_1}{\partial J_0^2}
\]

Evaluating some of these terms gives

\[
\frac{\partial H_1}{\partial J_0} = -\frac{mL^2 \omega_0^2}{24} \frac{1}{\theta_1^4} \frac{2J_0}{\pi^2 m^2 L^4 \omega_0^2} \overline{\sin^4 2\pi w_0} = -\frac{J_0}{32\pi^2 mL^2 \theta_1^2}
\]

\[
\frac{\partial H_1^2}{\partial J_0} = \frac{m^2 L^2 \omega_0^4}{576} \frac{1}{\theta_1^6} \frac{4J_0^3}{\pi^4 m^4 L^6 \omega_0^4} \overline{\sin^6 2\pi w_0} = \frac{35}{576} \frac{J_0^3}{32\pi^4 m^2 L^4 \theta_1^4}
\]

where we have used \(\overline{\sin^8 \phi} = 35/128\). This gives

\[
\alpha_2 = -\frac{J_0^3}{2048\pi^3 m^2 L^4 \theta_1^4 \omega_0}
\]

so

\[
\alpha = \frac{1}{2\pi} \omega_0 J_0 - \theta_1^2 \frac{J_0^2}{64\pi^2 mL^2 \theta_1^2} - \theta_1^4 \frac{J_0^3}{2048\pi^3 m^2 L^4 \theta_1^4 \omega_0}
\]

Finally we have

\[
\nu = \frac{\partial H}{\partial J_0} = \frac{\partial \alpha}{\partial J_0} = \frac{\omega_0}{2\pi} - \frac{J_0}{32\pi^2 mL^2} - \frac{3J_0^2}{2048\pi^3 m^2 L^4 \omega_0}
\]

Using \(J_0 = \pi mL^2 \omega_0 \theta_1^2\),

\[
\nu = \frac{\omega_0}{2\pi} \left( 1 - \frac{1}{16} \theta_1^2 - \frac{3}{1024} \theta_1^4 \right)
\]

The first order correction is the same as what we found from time dependent perturbation theory.