1. (a) Expanding out \( \eta \) in terms of its components we have

\[
\mathcal{L} = \frac{1}{2} \left[ \rho_0 (\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2) + 2P_0 \left( \frac{\partial \eta_1}{\partial x} + \frac{\partial \eta_2}{\partial y} + \frac{\partial \eta_3}{\partial z} \right) - \gamma P_0 \left( \frac{\partial \eta_1}{\partial x} + \frac{\partial \eta_2}{\partial y} + \frac{\partial \eta_3}{\partial z} \right)^2 \right].
\]

Then \( \pi_i = \partial \mathcal{L} / \partial \dot{\eta}_i = \rho_0 \dot{\eta}_i \), so \( \pi = (\rho_0 \dot{\eta}_1, \rho_0 \dot{\eta}_2, \rho_0 \dot{\eta}_3) \). The Hamiltonian density is therefore

\[
\mathcal{H} = \pi_1 \dot{\eta}_1 + \pi_2 \dot{\eta}_2 + \pi_3 \dot{\eta}_3 - \mathcal{L}
\]

\[
= \frac{1}{2} \rho_0 (\dot{\eta}_1^2 + \dot{\eta}_2^2 + \dot{\eta}_3^2) - P_0 \left( \frac{\partial \eta_1}{\partial x} + \frac{\partial \eta_2}{\partial y} + \frac{\partial \eta_3}{\partial z} \right) + \frac{1}{2} \gamma P_0 \left( \frac{\partial \eta_1}{\partial x} + \frac{\partial \eta_2}{\partial y} + \frac{\partial \eta_3}{\partial z} \right)^2.
\]

(b) The total Hamiltonian is the integral over space of the Hamiltonian density

\[
H = \int \mathcal{H} d^3x.
\]

Note that the divergence term (middle term in \( \mathcal{H} \)) becomes a surface term in the volume integral, and hence can be dropped. Then by doing an expansion

\[
\eta_n = \frac{1}{\sqrt{V}} \sum_k q_{nk} e^{ik \cdot r}
\]

\[
\pi_n = \frac{1}{\sqrt{V}} \sum_{k'} p_{nk'} e^{-ik' \cdot r}
\]

and using

\[
\frac{\partial \eta_n}{\partial x_j} = \frac{1}{\sqrt{V}} \sum_k ik_j q_{nk} e^{ik \cdot r}
\]

we get for \( H \)

\[
H = \sum_{k,k'} \frac{1}{V} \int \left[ \frac{1}{2 \rho_0} (p_{1k} p_{1k'} + p_{2k} p_{2k'} + p_{3k} p_{3k'}) e^{-i(k+k') \cdot r} - \frac{1}{2} \gamma P_0 (k \cdot q_k) (k' \cdot q_{k'}) e^{i(k+k') \cdot r} \right] d^3x.
\]

Since \( \int e^{-ir \cdot r} d^3x = V \delta_{a,0} \), then \( k' = -k \) in the sums. Also, since the \( \eta_j \) and \( \pi_j \) are real, we must have \( q_{-k} = q_k \) and \( p_{-k} = p_k \). Therefore we get

\[
H = \sum_k \left[ \frac{1}{2 \rho_0} (p_1^2 + p_2^2 + p_3^2) + \frac{1}{2} \gamma P_0 (k_1 q_1 + k_2 q_2 + k_3 q_3)^2 \right].
\]

2. (a) If \( G_i = -\int \sum_k \pi_k \partial \eta_k / \partial x_i \ d^3x \), then

\[
\frac{dG_i}{dt} = -\int \sum_k \left( \pi_k \frac{\partial \eta_k}{\partial x_i} + \pi_k \frac{\partial \dot{\eta}_k}{\partial x_i} \right) \ d^3x.
\]
Using Hamilton’s equations we get

\[
\frac{dG_i}{dt} = - \int \sum_k \left[ -\frac{\delta H}{\delta \eta_k} \frac{\partial \eta_k}{\partial x_i} + \pi_k \frac{\partial}{\partial x_i} \left( \frac{\delta H}{\delta \pi_k} \right) \right] d^3x.
\]

Using the definition of the functional derivative we get

\[
\frac{dG_i}{dt} = - \int \sum_k \left[ -\frac{\partial H}{\partial \eta_k} \frac{\partial \eta_k}{\partial x_i} + \sum_j \frac{d}{dx_j} \left( \frac{\partial H}{\partial \eta_{k,j}} \right) \frac{\partial \eta_k}{\partial x_i} + \pi_k \frac{\partial}{\partial x_i} \left( \frac{\partial H}{\partial \pi_k} \right) \right] d^3x.
\]

Integrating the second and third terms by parts, and throwing away the surface term that arises from the divergence piece that comes from the second term, we get

\[
\frac{dG_i}{dt} = - \int \sum_k \left[ -\frac{\partial H}{\partial \eta_k} \frac{\partial \eta_k}{\partial x_i} - \sum_j \frac{\partial H}{\partial \eta_{k,j}} \frac{\partial \eta_k}{\partial x_i x_j} + \frac{\partial}{\partial x_i} \left( \pi_k \frac{\partial H}{\partial \pi_k} \right) - \frac{\partial \pi_k}{\partial x_i} \frac{\partial H}{\partial \pi_k} \right] d^3x.
\]

Since \(\partial H/\partial \pi_k = \dot{\eta}_k\), the third term is simply \(\partial (H + \mathcal{L})/\partial x_i\). The remaining terms combine to give simply \(-\partial H/\partial x_i\). Therefore

\[
\frac{dG_i}{dt} = - \int \left[ -\frac{\partial H}{\partial x_i} + \frac{\partial}{\partial x_i} (H + \mathcal{L}) \right] d^3x = - \int \frac{\partial \mathcal{L}}{\partial x_i} d^3x.
\]

If \(H\) is not an explicit function of position, then \(\mathcal{L}\) won’t be either, and therefore \(dG_i/dt = 0\).

(b) The momentum density from the stress-energy tensor is

\[
p_i = T_i^0 = - \sum_\rho \frac{\partial \mathcal{L}}{\partial \eta_{\rho,i}} \eta_{\rho,i} + \mathcal{L} \delta_{i,0}.
\]

The last term vanishes, and since \(\partial \mathcal{L}/\partial \eta_{\rho} = \pi_{\rho}\), then integrating the momentum density gives

\[
P_i = - \int \left( \sum_\rho \pi_{\rho, i} \right) d^3x
\]

which is the same as \(G_i\) (since the dummy index \(\rho\) can be replaced by \(k\)).