Physics 362 - HOMEWORK SET #7

Solutions

1. (a) The constants of the motion can be determined from the initial values of the angles and their time derivatives: \( p_\phi = I_3[\dot{\psi}(0) + \dot{\phi}(0) \cos \theta_0] = I_3 \omega_0 \), \( p_\psi = I_1 \dot{\phi}(0) \sin^2 \theta_0 + I_3[\dot{\psi}(0) + \dot{\phi}(0) \cos \theta_0] \cos \theta_0 = I_3 \omega_0 \cos \theta_0 \) and \( E' = [I_1 \dot{\theta}(0)^2 + (p_\phi - p_\psi \cos \theta_0)^2/(I_1 \sin^2 \theta_0)]/2 = 0 \). Since energy must be conserved, the general expression for all time is

\[
0 = E' = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{(p_\phi - p_\psi \cos \theta)^2}{2 I_1 \sin^2 \theta}
\]

from which we can deduce that \( \dot{\theta} = 0 \) and \( p_\phi = p_\psi \cos \theta \) (i.e., \( \cos \theta = p_\phi/p_\psi = \cos \theta_0 \)). Both of these results imply \( \theta = \theta_0 \). We also have

\[
\dot{\phi} = \frac{p_\phi - p_\psi \cos \theta}{I_1 \sin^2 \theta} = \frac{p_\phi - p_\psi \cos \theta_0}{I_1 \sin^2 \theta_0} = 0
\]

so \( \phi(t) = \phi_0 \), and

\[
\dot{\psi} = \frac{p_\psi}{I_3} + \frac{(p_\psi \cos \theta - p_\phi) \cos \theta}{I_1 \sin^2 \theta} = \frac{p_\psi}{I_3} + \frac{(p_\psi \cos \theta_0 - p_\phi) \cos \theta}{I_1 \sin^2 \theta_0} = \omega_0
\]

which implies \( \psi(t) = \psi_0 + \omega_0 t \). Since \( \theta \) and \( \phi \) are constant, the object’s 3rd axis remains pointing in the same direction, and it rotates with constant angular velocity \( \omega_0 \) about this axis (\( \omega_3 = \dot{\psi} + \dot{\phi} \cos \theta = \omega_0 \)).

(b) In this case \( p_\psi = I_3[\dot{\psi}(0) + \dot{\phi}(0) \cos \theta_0] = I_3 \omega_3(0) = 0 \), \( p_\phi = I_1 \dot{\phi}(0) \sin^2 \theta_0 + p_\psi \cos \theta_0 = I_1 \omega_0 \sin^2 \theta_0 \), and \( E' = [I_1 \dot{\theta}(0)^2 + (p_\phi - p_\psi \cos \theta_0)^2/(I_1 \sin^2 \theta_0)]/2 = I_1 \omega_0^2 \sin^2 \theta_0/2 \). The energy conservation equation is

\[
E' = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{(p_\phi - p_\psi \cos \theta)^2}{2 I_1 \sin^2 \theta}
\]

so

\[
\frac{1}{2} I_1 \omega_0^2 \sin^2 \theta_0 = \frac{1}{2} I_1 \dot{\theta}^2 + \frac{1}{2} I_1 \omega_0^2 \sin^2 \theta_0
\]

\[
\frac{d\theta}{dt} = \dot{\theta} = \omega_0 \sin \theta_0 \sqrt{1 - \frac{\sin^2 \theta_0}{\sin^2 \theta}}
\]

\[
\int_{\theta_0}^{\theta} \frac{\sin \theta \, d\theta}{\sqrt{\sin^2 \theta - \sin^2 \theta_0}} = \int_{0}^{t} \omega_0 \sin \theta_0 \, dt
\]

\[-\sin^{-1} \left( \frac{\cos \theta}{\cos \theta_0} \right) \bigg|_{\theta_0}^{\theta} = \omega_0 \sin \theta_0 \cdot t
\]

Taking the cosine of both sides and using \( \sin^{-1}(1) = \pi/2 \) we get

\[
\cos \left[ \frac{\pi}{2} - \sin^{-1} \left( \frac{\cos \theta}{\cos \theta_0} \right) \right] = \cos(\omega_0 \sin \theta_0 \cdot t)
\]

\[
\sin \left[ \sin^{-1} \left( \frac{\cos \theta}{\cos \theta_0} \right) \right] = \cos(\omega_0 \sin \theta_0 \cdot t)
\]

\[
\frac{\cos \theta}{\cos \theta_0} = \cos(\omega_0 \sin \theta_0 \cdot t)
\]

\[
\cos \theta(t) = \cos \theta_0 \cos(\omega_0 \sin \theta_0 \cdot t).
\]
Next we have (defining $\Omega = \omega_0 \sin \theta_0$)

$$\dot{\phi} = \frac{p_\phi - p_\theta}{I_1 \sin^2 \theta} = \frac{I_1 \omega_0 \sin^2 \theta_0}{I_1 \sin^2 \theta} = \frac{\omega_0 \sin^2 \theta_0}{\sin^2 \theta}$$

$$\int_{\phi_0}^{\phi} d\phi = \int_{0}^{t} \frac{\omega_0 \sin^2 \theta_0}{1 - \cos^2 \theta} \cos \Omega t \cos \Omega t \, dt = \int_{0}^{t} \frac{\omega_0 \sin^2 \theta_0}{1 - \cos^2 \theta_0 \cos^2 \Omega t} \, dt$$

$$\phi(t) - \phi_0 = \omega_0 \sin^2 \theta_0 \left[ \frac{1}{\Omega \sin \theta_0} \tan^{-1} \left( \frac{\tan \Omega t}{\sin \theta_0} \right) \right]_{0}^{t}$$

$$\phi(t) = \frac{\omega_0 \sin^2 \theta_0}{\Omega \sin \theta_0} \tan^{-1} \left( \frac{\tan \Omega t}{\sin \theta_0} \right) = \tan^{-1} \left( \frac{\tan \Omega t}{\sin \theta_0} \right)$$

$$\tan \phi(t) = \frac{\tan(\omega_0 \sin \theta_0 t)}{\sin \theta_0}$$

where the last two steps used $\Omega = \omega_0 \sin \theta_0$. Finally, we also have

$$\dot{\psi} = \frac{p_\psi}{I_3} + \left( \frac{p_\psi \cos \theta - p_\theta}{I_1 \sin^2 \theta} \right) \cos \theta = \frac{-p_\phi \cos \theta}{I_1 \sin^2 \theta} = \frac{-I_1 \omega_0 \sin^2 \theta_0 \cos \theta}{I_1 \sin^2 \theta}$$

$$\psi(t) - \psi_0 = -\omega_0 \sin^2 \theta_0 \int_{0}^{t} \frac{\cos \theta_0 \cos \Omega t}{1 - \cos^2 \theta_0 \cos^2 \Omega t} \, dt$$

$$\psi(t) = -\omega_0 \sin^2 \theta_0 \frac{1}{\Omega \sin \theta_0} \tan^{-1} \left( \frac{\cos \theta_0 \sin \Omega t}{\sin \theta_0} \right) = -\tan^{-1} \left( \frac{\cos \theta_0 \sin \Omega t}{\sin \theta_0} \right)$$

$$\tan \psi(t) = \frac{-\cos \theta_0}{\sin \theta_0} \tan(\omega_0 \sin \theta_0 t).$$

For $\theta_0 = \pi/2$, we get $\sin \theta_0 = 1$ and $\cos \theta_0 = 0$, so $\cos \theta(t) = 0$, $\tan \phi(t) = \tan \omega_0 t$ and $\tan \psi(t) = 0$. Therefore $\theta(t) = \pi/2$, $\psi(t) = 0$ (i.e., both are constant), and $\phi(t) = \omega_0 t$, and the body rotates around in the horizontal plane with constant angular velocity (like a bottle spinning on a table); the body is not spinning about its symmetry axis since $\omega_3 = \dot{\psi} + \dot{\phi} \cos \theta = 0$.

For $\theta_0 \neq \pi/2$, $\theta$ will oscillate back and forth about $\pi/2$. Since $\dot{\phi} = \omega_0 \sin^2 \theta_0 \sin^2 \theta$, the object rotates around (precesses) at a varying rate (because $\theta$ is not constant). As usual, the rotation about the symmetry axis in the body frame (given by $\omega_3$) is constant.

2. (a) The effective potential is the same as the symmetrical top except that the gravity term has the opposite sign, so following from Eq. 11.56 in the text we get

$$V_{eff}(\theta) = \frac{(p_\theta - p_\psi \cos \theta)^2}{2I_1 \sin^2 \theta} - MgL \cos \theta.$$

To find the equilibrium point we take the derivative and set equal to zero (a quick way is to use Eq. 11.60 in the text and reverse the sign of the gravity term):

$$0 = \frac{\partial V_{eff}}{\partial \theta} = MgL \sin \theta + \frac{(p_\theta - p_\psi \cos \theta)(p_\psi - p_\theta \cos \theta)}{I_1 \sin^2 \theta}.$$

(b) For small $\theta$, $\cos \theta \approx 1$ and $\sin \theta \approx \theta$, so

$$MgL \theta^4 \approx \frac{(p_\theta - p_\psi)^2}{I_1} \quad \Rightarrow \quad \theta_{eq} \approx \sqrt[4]{\frac{|p_\theta - p_\psi|}{\sqrt{MgL I_1}}}.$$
Notice that $\theta$ will be small if $p_\phi, p_\psi \ll \sqrt{MgL_1}$.

c) To find the angular frequency of small oscillations about equilibrium we must find the second derivative of $V_{\text{eff}}$ evaluated at the equilibrium point. The second derivative can be written (after some algebra) as

$$\frac{\partial^2 V_{\text{eff}}}{\partial \theta^2} = MgL \cos \theta + \frac{(p_\phi^2 + p_\psi^2)(1 + 2 \cos^2 \theta) - p_\phi p_\psi(5 + \cos^2 \theta)}{I_1 \sin^4 \theta}.$$  

We could also have taken the limit $\theta \ll 1$ in $\partial V_{\text{eff}}/\partial \theta$ before taking the second derivative (but not before taking the first derivative). When evaluated at the equilibrium point, $\cos \theta \simeq 1$ and $\sin^4 \theta \simeq \theta^4 \simeq (p_\phi - p_\psi)^2/(MgL_1)$, and

$$\frac{\partial^2 V_{\text{eff}}}{\partial \theta^2} = MgL + \frac{3p_\phi^2 - 6p_\phi p_\psi + 3p_\psi^2}{I_1(p_\phi - p_\psi)^2/(MgL_1)} = 4MgL.$$  

Then the angular frequency of small oscillations about equilibrium is $\omega = \sqrt{\frac{\partial^2 V_{\text{eff}}}{\partial \theta^2}}/I_1 = \sqrt{\frac{4MgL}{I_1}} = 2\sqrt{\frac{2MgL}{I_1}}$. This happens to be twice the angular frequency for small oscillations of the physical pendulum when it oscillates with the symmetry axis moving in a vertical plane (as opposed this case, where the symmetry axis rotates around in the approximate shape of a cone, except for the small oscillations in $\theta$).