Physics 362 - HOMEWORK SET #3
Solutions

1. We let the symmetry axis be the $z$ axis, with the tip of the cone at the origin, and let the $x$ axis be the rotation axis. The moment of inertia is $\int \rho(z^2 + y^2) dV$ since $z^2 + y^2$ is the square of the distance of a given point $(x, y, z)$ from the rotation axis (the $x$ axis). In cylindrical coordinates $y = r \sin \phi$, so

$$y^2 = r^2 \sin^2 \phi = r^2(1 - \cos 2\phi)/2,$$

and

$$I = \int_0^h \left[ \int_0^{r_{\text{max}}} \left( \int_0^{2\pi} \rho \left[ z^2 + r^2 \left( 1 - \cos 2\phi \right)/2 \right] d\phi \right) r \, dr \right] dz = 2\pi \rho \int_0^h \left[ \int_0^{Rz/h} \left[ z^2 + r^3/2 \right] dr \right] dz$$

where the $\cos 2\phi$ part of the integral vanishes when evaluated between the limits 0 and $2\pi$. Then

$$I = 2\pi \rho \int_0^h \left[ \int_0^{Rz/h} \left( \frac{1}{2} r^2 z^2 + \frac{1}{8} r^4 \right) \right] \, dz = 2\pi \rho \int_0^h \left( \frac{1}{2} \frac{R^2 z^2}{h^2} z^2 + \frac{1}{8} \frac{R^4 z^4}{h^4} \right) \, dz$$

$$= 2\pi \rho \left( \frac{R^2}{2h^2} + \frac{R^4}{8h^4} \right) \int_0^h z^4 \, dz = \frac{\pi \rho R^2}{4h^4} (4h^2 + R^2) \frac{1}{5} h^5 = \frac{\pi}{20} \rho R^2 h (4h^2 + R^2).$$

The mass $M$ is $\int \rho dV$, or

$$M = \int_0^h \left[ \int_0^{r_{\text{max}}} \left( \int_0^{2\pi} \rho \, d\phi \right) r \, dr \right] dz = 2\pi \rho \int_0^h \left[ \int_0^{Rz/h} r \, dr \right] dz$$

$$= 2\pi \rho \int_0^h \left( \frac{1}{2} r^2 \right) \, dz = 2\pi \rho \int_0^h \frac{1}{2} \frac{R^2 z^2}{h^2} \, dz$$

$$= 2\pi \rho R^2 \left( \frac{1}{3} z^3 \right) \bigg|_0^h = \frac{\pi \rho R^2}{h^2} \frac{1}{3} h^3 = \frac{\pi}{3} \rho R^2 h.$$

Therefore

$$\frac{I}{M} = \frac{\pi \rho R^2 h (4h^2 + R^2)}{20} \frac{3}{\pi \rho R^2 h} = \frac{3}{20} (4h^2 + R^2)$$

$$I = \frac{3}{20} M (4h^2 + R^2) = Mk^2 \quad \implies \quad k = \sqrt{\frac{3}{20} (4h^2 + R^2)}.$$

2. From the reflection symmetry about the $y$ axis the $x$ position of the center of mass must be 0. For the $y$ position we use $y_{CM} = \int \rho y \, ds/M$, where $M = \int \rho \, ds$; the integral is over the length of the wire. In this case we can use polar coordinates in the $x$-$y$ plane, so $ds = R \, d\theta$ and

$$y_{CM} = \frac{\int_0^\pi \rho y \, R \, d\theta}{\int_0^\pi \rho \, R \, d\theta} = \frac{\int_0^\pi y \, d\theta}{\int_0^\pi d\theta} = \frac{\int_0^\pi (R \sin \theta) \, d\theta}{\pi} = \frac{R}{\pi} \int_0^\pi \sin \theta \, d\theta$$

$$= \frac{R}{\pi} (-\cos \theta) \bigg|_0^\pi = \frac{2R}{\pi}.$$

The moment of inertia of a full circular ring about an axis in the plane of the ring that goes through the center of the ring is $MR^2/2$ (see class notes or page 226 of the text). Since the semicircular ring has the same horizontal mass distribution (only the bottom half is missing), it has the same moment of inertia formula about the $y$ axis, i.e., $I = MR^2/2$ (it has half the moment of inertia but also half the mass of a
full ring, so in terms of its mass \( M \) the formula for \( I \) is the same). Since the \( y \) axis goes through the center of mass, this is the answer for the axis parallel to the \( y \) axis (it actually is the \( y \) axis); \( I_{CM,y} = MR^2/2 \).

The moment of inertia for rotation about the \( x \) axis of the semicircle is likewise \( MR^2/2 \). However, the axis parallel to the \( x \) axis that goes through the center of mass is a distance \( 2R/\pi \) away from the \( x \) axis since the position of the center of mass is \( 2R/\pi \) above the \( x \) axis. Using the parallel axis theorem, the moment of inertia of the axis parallel to the \( x \) axis that goes through the center of mass, \( I_{CM,x} \) is given by

\[
\frac{1}{2} MR^2 = I_{CM,x} + M \left( \frac{2R}{\pi} \right)^2
\]

\[
I_{CM,x} = MR^2 \left( \frac{1}{2} - \frac{4}{\pi^2} \right).
\]

Finally, using the perpendicular axis theorem,

\[
I_{CM,z} = I_{CM,x} + I_{CM,y} = MR^2 \left( \frac{1}{2} - \frac{4}{\pi^2} \right) + \frac{1}{2} MR^2
\]

\[
= MR^2 \left( 1 - \frac{4}{\pi^2} \right).
\]

3. (a) The net force acting on the object is \( 10 \text{ N}\hat{x} - 2 \text{ N}\hat{x} + 2 \text{ N}\hat{y} + 6 \text{ N}\hat{y} = 8 \text{ N}\hat{x} + 8 \text{ N}\hat{y} \). Since this is not zero, this is the resultant. It has magnitude \( 8\sqrt{2} \text{ N} \) and acts at a \( 45^\circ \) angle to the \( x \) axis. To determine where it must act, we calculate the torque about the center of mass (i.e., about the center of the square).

Each force acts along an edge, which is \( 20 \text{ cm} = 0.20 \text{ m} \) from the center. Using the convention that positive torque is in the counter-clockwise direction, we find

\[
\tau = -(10 \text{ N})(0.20 \text{ m}) + (2 \text{ N})(0.20 \text{ m}) - (2 \text{ N})(0.20 \text{ m}) + (6 \text{ N})(0.20 \text{ m})
\]

\[
= -0.8 \text{ N} \cdot \text{m}.
\]

This is negative, and so must be in the clockwise direction. Since the torque is \( \tau = Fr_\perp \), then the perpendicular distance from the line of action of the force to the center of mass is \( r_\perp = \tau/F = 0.8/(8\sqrt{2}) \text{ m} = 1/(10\sqrt{2}) \text{ m} = \sqrt{2}/20 \text{ m} \). Since the force acts at \( 45^\circ \) to the \( x \) axis, the perpendicular distance makes an angle of \( 45^\circ \) with respect to the \(-x\) axis (see diagram).

(b) The net force acting on the object is \(-2 \text{ N}\hat{x} - 2 \text{ N}\hat{y} + (2\sqrt{2} \text{ N}\cos 45^\circ \hat{x} + 2\sqrt{2} \text{ N}\sin 45^\circ \hat{y}) = 0 \). Hence the net result can be only a couple or zero, depending on whether or not there is a net torque. To calculate the torque about the center of mass, we first must identify the center of mass. From reflection symmetry about a vertical axis through the center, the center of mass must lie down the middle of the long center piece. The object can be thought of as two objects, both 6 m by 2 m: their centers of mass are at 1 m above the base and 5 m above the base. Since they have equal masses, the center of mass is at the average position, 3 m above the base.

The \(-2 \text{ N}\hat{x}\) force acts horizontally 6 m above the base, so its moment arm is 3 m and it has torque \( \tau = Fr_\perp = (2 \text{ N})(3 \text{ m}) = 6 \text{ N} \cdot \text{m} \); it is positive since the torque causes counter-clockwise rotation. The \(-2 \text{ N}\hat{y}\) force acts vertically 3 m from the center, so it has torque \( \tau = Fr_\perp = (2 \text{ N})(3 \text{ m}) = 6 \text{ N} \cdot \text{m}, also positive (counter-clockwise). The other force acts at a point that is 3 m to the right and 3 m below the center of mass, so the distance from the center of mass is \( 3\sqrt{2} \text{ m} \). Since the point of action relative to the
center of mass is at a $-45^\circ$ angle and the force acts at a $45^\circ$ angle, $3\sqrt{2} \text{ m}$ is the perpendicular distance (moment arm), and $\tau = F r_\perp = (2\sqrt{2} \text{ N})(3\sqrt{2} \text{ m}) = 12 \text{ N}\cdot\text{m}$, also positive. The total torque is therefore $24 \text{ N}\cdot\text{m}$. One example of a couple that could produce this torque is two $4 \text{ N}$ forces, one acting down at the lower left corner and the other acting up at the lower right corner; each has a line of action with perpendicular distance $3 \text{ m}$ from the center of mass.