Physics 362 - HOMEWORK SET #11
Solutions

1. (a) The electrons have 4-momenta \((E_1, E_1, 0, 0)\) and the positrons have 4-momenta \((E_2, -E_2, 0, 0)\) (if we ignore their masses). Then the total momentum is \((E_1 + E_2, E_1 - E_2, 0, 0)\), and the invariant mass (or center of mass energy) squared is \(M^2 = (E_1 + E_2)^2 - (E_1 - E_2)^2 = 4E_1E_2\). To create a \(\Upsilon\) at rest in the CM, we must have \(M = 10.58\ \text{GeV}\), so \(E_2 = M^2/(4E_1) = (10.58)^2/4/9.00 = 3.11\ \text{GeV}\) for the positrons. The total momentum is then \(p_{\text{tot}} = (9.00, 9.00, 0, 0) + (3.11, -3.11, 0, 0) = (12.11, 5.89, 0, 0)\); applying a Lorentz transformation gives

\[
\begin{pmatrix}
\gamma & -\gamma \beta & \gamma (12.11 - 5.89 \beta) & \gamma (5.89 - 12.11 \beta)
\end{pmatrix}
\begin{pmatrix}
12.11 \\
5.89 \\
0
\end{pmatrix}
\begin{pmatrix}
M \\
0
\end{pmatrix}
\]

so that \(\beta = 5.89/12.11 = 0.4864\). Therefore the velocity of the CM is \(0.4864\ c\) in the \(+x\) direction.

(b) Given \(\beta = 0.4864\), then \(\gamma = 1.1445\) and \(\gamma \beta = 0.55665\). A \(B\) meson has mass 5.28, and therefore in the \(\Upsilon\) rest frame has spatial momentum \(p = \sqrt{m_Y^2 - 4m_B^2/2} = 0.325\ \text{GeV}\) and energy \((m_Y^2 + m_B^2 - m_Y^2)/(2m_Y) = m_Y/2 = 5.29\ \text{GeV}\). Therefore in the case where the decay products move along the \(x\) direction, the 4-momenta of the \(B\) and \(\bar{B}\) are \((5.29, 0.325, 0, 0)\) and \(5.29, -0.325, 0, 0\) in the CM frame. Boosting to the lab frame, the 4-momentum of the \(B\) is (considering only the time and \(x\) components)

\[
\begin{pmatrix}
E \\
p_x
\end{pmatrix}
= \begin{pmatrix}
\gamma & \gamma \beta \\
\gamma \beta & \gamma
\end{pmatrix}
\begin{pmatrix}
5.29 \\
0.325
\end{pmatrix}
\begin{pmatrix}
1.1445 \\
0.55665
\end{pmatrix}
\begin{pmatrix}
0.325
\end{pmatrix}
= \begin{pmatrix}
6.24 \\
3.32
\end{pmatrix}\ \text{GeV}
\]

where the Lorentz transformation has \(+\gamma \beta\) instead of \(-\gamma \beta\) since we are transforming from the CM to the lab and not vice versa (the lab is moving backwards relative to the CM). For the \(\bar{B}\),

\[
\begin{pmatrix}
E \\
p_x
\end{pmatrix}
= \begin{pmatrix}
\gamma & \gamma \beta \\
\gamma \beta & \gamma
\end{pmatrix}
\begin{pmatrix}
5.29 \\
-0.325
\end{pmatrix}
\begin{pmatrix}
1.1445 \\
0.55665
\end{pmatrix}
\begin{pmatrix}
0.325
\end{pmatrix}
= \begin{pmatrix}
5.87 \\
2.57
\end{pmatrix}\ \text{GeV}.
\]

Notice that both particles are moving in the \(+x\) direction in the lab frame. Also, the total energy is 12.11 GeV and the total momentum is 5.89 GeV, as they should be.

For the case where both decay products go in the \(y\) direction in the CM, their 4-momenta are \((5.29, 0, 0.325, 0)\) and \(5.29, 0, -0.325, 0\) in the CM frame. Boosting to the lab frame, the 4-momentum of the \(B\) is (considering only the time, \(x\) and \(y\) components)

\[
\begin{pmatrix}
E \\
p_x \\
p_y
\end{pmatrix}
= \begin{pmatrix}
\gamma & \gamma \beta & 0 \\
\gamma \beta & \gamma & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
5.29 \\
0 \\
0.325
\end{pmatrix}
\begin{pmatrix}
1.1445 \\
0.55665 \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\begin{pmatrix}
5.29 \\
0 \\
0.325
\end{pmatrix}
\begin{pmatrix}
6.0554 \\
2.944 \\
0.325
\end{pmatrix}\ \text{GeV}
\]

For the \(\bar{B}\),

\[
\begin{pmatrix}
E \\
p_x \\
p_y
\end{pmatrix}
= \begin{pmatrix}
\gamma & \gamma \beta & 0 \\
\gamma \beta & \gamma & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
5.29 \\
0 \\
-0.325
\end{pmatrix}
\begin{pmatrix}
1.1445 \\
0.55665 \\
0
\end{pmatrix}
\begin{pmatrix}
0 \\
0 \\
1
\end{pmatrix}
\begin{pmatrix}
5.29 \\
0 \\
-0.325
\end{pmatrix}
\begin{pmatrix}
6.0554 \\
2.944 \\
-0.325
\end{pmatrix}\ \text{GeV}.
\]
Notice that both particles are at a small angle from the $+x$ axis in the lab frame ($\theta = \arctan(0.325/2.9439) \approx 6.3^\circ$). Also, the total energy is 12.11 GeV and the total momentum is 5.89 GeV, as they should be.

2. (a) The initial energy and momentum of the electron is $(E_1, p_1)$, where $p_1 = \sqrt{E_1^2 - m^2}$, and for the photon is $(E_2, -E_2)$. Let the final values be $(E_3, p_3)$ and $(E_4, E_4)$. Then conservation of energy gives $E_1 + E_2 = E_3 + E_4$, and conservation of momentum gives $p_1 - E_2 = p_3 + E_4$. Note that $p_1$ is positive because of the direction of the initial electron, but we have not constrained the sign of $p_3$ (i.e., it could be negative). Then

$$E_3^2 - m^2 = p_3^2 = (p_1 - E_2 - E_4)^2$$

$$(E_1 + E_2 - E_4)^2 - m^2 = (p_1 - E_2 - E_4)^2$$

$$E_1^2 + E_2^2 + E_4^2 + 2E_1E_2 - 2E_1E_4 - 2E_2E_4 - m^2 = p_1^2 + E_2^2 + E_4^2 - 2p_1E_2 - 2p_1E_4 + 2E_2E_4$$

$$2E_1E_2 - 2E_1E_4 - 2E_2E_4 = -2p_1E_2 - 2p_1E_4 + 2E_2E_4$$

where we have used $E_3^2 - m^2 = p_3^2$. Solving for $E_4$ gives

$$E_4 = \frac{E_2(E_1 + p_1)}{2E_2 + E_1 - p_1} = \frac{E_2(E_1 + \sqrt{E_1^2 - m^2})}{2E_2 + E_1 - \sqrt{E_1^2 - m^2}}.$$

Then

$$E_3 = E_1 + E_2 - E_4 = \frac{2E_2(E_1 + E_2 - \sqrt{E_1^2 - m^2}) + E_1(E_1 - \sqrt{E_1^2 - m^2})}{2E_2 + E_1 - \sqrt{E_1^2 - m^2}}$$

and

$$p_3 = p_1 - E_2 - E_4 = \frac{2E_2(\sqrt{E_1^2 - m^2} - E_1 - E_2) + \sqrt{E_1^2 - m^2}(E_1 - \sqrt{E_1^2 - m^2})}{2E_2 + E_1 - \sqrt{E_1^2 - m^2}}.$$  

(b) In the limit $m \ll E_1$ we can drop $m$ from the above expression and

$$E_4 = \frac{2E_2E_1}{2E_2 + E_1} = E_1$$

so the energy of the outgoing photon is the same as the energy of the incoming electron. Also, since $p_1 = E_1$ when the mass is ignored, the momentum of the outgoing photon $p_4 = E_4$ is the same as the momentum of the incoming electron $p_1 = E_1$. Furthermore, $E_3 = E_1 + E_2 - E_4 = E_2$ and the energy of the outgoing electron is the same as the energy of the incoming photon. Finally, $p_3 = p_1 - E_2 - E_4 = E_1 - E_2 - E_3 = -E_2$, and the momentum of the outgoing electron is the same as that of the incoming photon.

(c) For $p_1 = 0$ we get $E_1 = m$ and

$$E_4 = \frac{E_2E_1}{2E_2 + E_1} = \frac{E_2m}{2E_2 + m} = \frac{E_2}{1 + 2E_2/m}.$$  

The corresponding expression for the Compton effect with scattering angle $\pi$ is

$$E_4 = \frac{E_2}{1 + E_2(1 - \cos \theta)/m} = \frac{E_2}{1 + 2E_2/m}$$

for $\theta = \pi$. 