Physics 362 - HOMEWORK SET #1
Solutions

1. The fluid is incompressible if \( \nabla \cdot \mathbf{v} = 0 \). It is irrotational if \( \nabla \times \mathbf{v} = 0 \).

(a) \( \nabla \cdot \mathbf{v} = \partial v_x/\partial x + \partial v_y/\partial y + \partial v_z/\partial z = 2ax - 2ax + 0 = 0 \), so it is incompressible. Next we check

\[
\nabla \times \mathbf{v} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\partial/\partial x & \partial/\partial y & \partial/\partial z \\
a(x^2 - y^2) & -2axy & 0
\end{vmatrix} = (-2ay + 2ay) \hat{z} = 0
\]

so it is also irrotational. The velocity potential function can be found by taking a line integral from a reference point (which I will choose to be the origin) to a general point \((x, y, z)\). Any path will work; I will choose the path that goes from \((0, 0, 0)\) to \((x, 0, 0)\), then from \((x, 0, 0)\) to \((x, y, 0)\), and finally from \((x, y, 0)\) to \((x, y, z)\):

\[
\phi = \int_{(x,0,0)}^{(x,y,0)} \mathbf{v} \cdot d\mathbf{r} + \int_{(x,y,0)}^{(x,y,z)} \mathbf{v} \cdot d\mathbf{r} + \int_{(x,y,z)}^{(x,y,0)} \mathbf{v} \cdot d\mathbf{r}
\]

\[
= \int_0^x v_x dx + \int_0^y v_y dy \bigg|_{x=x,z=0} + \int_0^z v_z dz \bigg|_{x=x,y=y}
\]

\[
= \int_0^x ax^2 dx + \int_0^y (-2axy) dy + \int_0^z (0) dz
\]

\[
= \frac{1}{3}ax^3 - axy^2.
\]

(b) \( \nabla \cdot \mathbf{v} = \partial v_x/\partial x + \partial v_y/\partial y + \partial v_z/\partial z = 6ax - 4ax - 2ax = 0 \), so it is incompressible. Next we check

\[
\nabla \times \mathbf{v} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\partial/\partial x & \partial/\partial y & \partial/\partial z \\
a^2x + bxy & cyx + dxy^2 & 0
\end{vmatrix} = (cy - bx) \hat{z}
\]

so it is rotational. The corresponding angular velocity is \( \omega = \frac{1}{2} \nabla \times \mathbf{v} = az \hat{y} - 2ay \hat{z} \).

2. (a) The field satisfies the continuity equation if \( \partial \rho/\partial t + \nabla \cdot (\rho \mathbf{v}) = 0 \). For an incompressible fluid, \( \rho \) is constant and this reduces to \( \nabla \cdot \mathbf{v} = 0 \), the usual incompressibility condition. In this case \( \nabla \cdot \mathbf{v} = \partial v_x/\partial x + \partial v_y/\partial y + \partial v_z/\partial z = (2ax + by) + (cx + 2dy) + 0 = 0 \), so it is incompressible if \( 2a + c = 0 \) and \( b + 2d = 0 \), i.e., the \( x \) term and the \( y \) term must vanish separately if this is to be zero at all points in space.

(b) The fluid is irrotational if \( \nabla \times \mathbf{v} = 0 \):

\[
\nabla \times \mathbf{v} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\partial/\partial x & \partial/\partial y & \partial/\partial z \\
ax^2 + bxy & cyx + dxy^2 & 0
\end{vmatrix} = (cy - bx) \hat{z}
\]

This can be zero only if \( c = 0 \) and \( b = 0 \) (again, the \( x \) and \( y \) terms must vanish separately if it is to be zero at all points in space). When combined with the incompressibility condition, \( \mathbf{v} = 0 \) is the only possibility of this form that can be both incompressible and irrotational.
3. The potential energy is

\[ u = \int_{P_0}^{P} \frac{P}{\rho B} \, dP. \]

Using \( B = P \) and \( P = C \rho \), we get

\[ u = \int_{P_0}^{P} \frac{C}{B} \, dP = C \ln(P/P_0) = \frac{P}{\rho} \ln(P/P_0). \]

4. The continuity equation for steady flow is \( \rho_1 v_1 A_1 = \rho_2 v_2 A_2 \), and Bernoulli's equation for steady flow is \( \frac{1}{2} v_1^2 + (P_1/\rho_1) - \mathcal{G}_1 + u_1 = \frac{1}{2} v_2^2 + (P_2/\rho_2) - \mathcal{G}_2 + u_2 \). In this case the flow is horizontal, so the gravitational potential term may be dropped, since it is the same on both sides.

(a) For an incompressible fluid, \( \rho_2 = \rho_1 \), so

\[ \rho_2 = 2.5 \text{ kg/m}^3 \]

and the continuity equation for steady flow then gives

\[ A_2 = A_1 v_1/v_2 = (0.10)(50)/(100) = 0.050 \text{ m}^2. \]

Also, \( u = 0 \) for an incompressible fluid, so Bernoulli’s equation gives

\[ P_2 = P_1 + \frac{1}{2} \rho(v_1^2 - v_2^2) = 2 \times 10^5 + \frac{1}{2}(2.5)(50^2 - 100^2) = 1.906 \times 10^5 \text{ N/m}^2. \]

(b) An ideal adiabatic gas has \( P = C \rho^\gamma \), so

\[ P_1\rho_1^\gamma = P_2\rho_2^\gamma. \]

The continuity equation for steady flow is

\[ \rho_1 v_1 A_1 = \rho_2 v_2 A_2. \]

From the class notes, \( u = P/[(\rho(\gamma - 1)] \) and Bernoulli’s equation can be written

\[ \frac{1}{2} v_1^2 + \frac{\gamma}{\gamma - 1} \frac{P_1}{\rho_1} = \frac{1}{2} v_2^2 + \frac{\gamma}{\gamma - 1} \frac{P_2}{\rho_2}. \]

From these three equations we can solve for \( P_2, \rho_2 \) and \( A_2 \). First, since \( \gamma = 5/3 \), we have \( \gamma/(\gamma - 1) = 5/2 \). Then using the first equation to substitute in for \( P_2 \) in the third equation we get

\[ \frac{1}{2} v_1^2 + \frac{5}{2} \frac{P_1}{\rho_1} = \frac{1}{2} v_2^2 + \frac{5}{2} \frac{1}{\rho_2} \frac{P_1\rho_2^{5/3}}{\rho_1^{5/3}} \]

or, solving for \( \rho_2 \),

\[ \rho_2^{2/3} = \rho_1^{2/3} + \frac{\rho_1^{5/3}}{5P_1}(v_1^2 - v_2^2) \]

\[ \rho_2^{2/3} = (2.5)^{2/3} + \frac{(2.5)^{5/3}}{5(2 \times 10^5)}(50^2 - 100^2) \]

\[ \rho_2^{2/3} = 1.8075 \]

\[ \rho_2 = (1.8075)^{3/2} = 2.43 \text{ kg/m}^3. \]
Then
$$P_2 = P_1\left(\frac{\rho_2}{\rho_1}\right)^{5/3} = 2 \times 10^5 \left(\frac{2.43}{2.5}\right)^{5/3} = 1.908 \text{ N/m}^2$$
and
$$A_2 = A_1\frac{\rho_1 v_1}{\rho_2 v_2} = (0.10)\frac{(2.5)(50)}{(2.43)(100)} = 0.0514 \text{ m}^2.$$  

(c) To find the Mach number, we first must calculate the speed of sound in a gas, which is given by
$$v_s = \sqrt{B/\rho},$$
where $B = \gamma P$ for an adiabatic situation. At the end with the largest speed
$$M = \frac{v}{v_s} = v \sqrt{\frac{\rho}{\gamma P}} = (100) \sqrt{\frac{2.43}{(1.667)(2.0 \times 10^5)}} = 0.27.$$  
Since this is small but not negligible (i.e., less than but not much smaller than unity), we would expect that the adiabatic flow should be somewhat different from the incompressible case, but not greatly so; this is consistent with the answers to (a) and (b).