1. The Earth’s shape is approximately an ellipsoid of revolution, with principal moments $I_1 = I_2$ and $I_3 = (1 + \epsilon)I_1$, where $\epsilon \approx 0.00327$ (the third principal axis corresponds roughly to the axis of rotation, i.e., to a line running between the north and south poles). Assuming that the Earth is a rigid body and the angular velocity vector is not exactly aligned with the third principal axis, find the approximate precession period (in days) of the angular velocity vector about the third principal axis.

2. For this problem, let the Earth be considered a rigid solid sphere of mass $M$ and radius $R$, plus a mountain of mass $m = \epsilon M$ on the surface at $45^\circ$ latitude in the northern hemisphere, where $\epsilon \ll 1$. Let the line between the north and south poles be along the $z$ axis, assume that the mountain lies in the $y$-$z$ plane and that the system is initially rotating with angular velocity vector $\omega_0 \hat{z}$ (see diagram).

(a) Show that the inertia tensor can be written in this basis as

$$I = \frac{MR^2}{10} \begin{pmatrix} 4 + 10\epsilon & 0 & 0 \\ 0 & 4 + 5\epsilon & -5\epsilon \\ 0 & -5\epsilon & 4 + 5\epsilon \end{pmatrix}.$$

[Hint: you should be able to write down the part of the inertia tensor for the sphere without doing a calculation.]

(b) What is the initial angular momentum in this basis? To leading order in $\epsilon$, what is the magnitude of the angle between the initial angular momentum and the initial angular velocity? [Hint: since the initial angular velocity is in the $z$ direction, this is just the usual polar angle of the angular momentum.]

(c) Find the principal moments and principal axes of the inertia tensor in (a). [Hint: you should find that two of the principal moments are degenerate.]

(d) Find the matrix that represents the transformation from the original basis to the principal axis basis. What are the components of the initial angular velocity in the principal axis basis?

(e) Find the period of precession for the rotation axis about the third principal axis to leading order in $\epsilon$ (i.e., you may drop terms that are of order $\epsilon^2$ or higher). How long is this (in years) if $\epsilon = 10^{-9}$?