1. (a) The center of mass is given by $x_{CM} = \sum m_i x_i$, etc. Then

$$x_{CM} = \frac{m(a - a + 0)}{3m} = 0, \quad y_{CM} = \frac{m(-a - a + 2a)}{3m} = 0, \quad z_{CM} = \frac{0 + 0 + 0}{3m} = 0.$$ 

(b) The inertia tensor elements are

$$I_{xx} = \sum m_i (y_i^2 + z_i^2) = m \left[ (-a)^2 + 0 + (-a)^2 + 0 + (2a)^2 + 0 \right] = 6ma^2$$

$$I_{yy} = \sum m_i (x_i^2 + z_i^2) = m \left[ a^2 + 0 + (-a)^2 + 0 + 0 + 0 \right] = 2ma^2$$

$$I_{zz} = \sum m_i (x_i^2 + y_i^2) = m \left[ a^2 + (-a)^2 + (-a)^2 + (-a)^2 + 0 + (2a)^2 \right] = 8ma^2$$

$$I_{xy} = I_{yx} = -\sum m_i x_i y_i = -m [(a)(a) + (-a)(-a) + (0)(2a)] = 0$$

$$I_{xz} = I_{zx} = -\sum m_i x_i z_i = -m [(a)(0) + (-a)(0) + (0)(0)] = 0$$

$$I_{yz} = I_{zy} = \sum m_i y_i z_i = m [(-a)(0) + (-a)(0) + (2a)(0)] = 0.$$ 

(c) Since the inertia tensor is diagonal, the principal axes are the $x$, $y$ and $z$ axes, with principal moments of inertia given by the diagonal elements $6ma^2$, $2ma^2$ and $8ma^2$. Note that $I_{xx} + I_{yy} = I_{zz}$, as required by the perpendicular axis theorem. 

(d) According to the tennis racket theorem, rotation is stable about the principal axes with the largest and smallest moments, in this case the $y$ and $z$ axes. It is unstable about the axis with the middle moment, i.e., the $x$ axis.

2. (a) Euler’s equations are (using $I_1 = I_2$)

$$I_1 \ddot{\omega}_1 + (I_3 - I_1)\omega_2 \omega_3 = \tau_0$$

$$I_1 \ddot{\omega}_2 + (I_1 - I_3)\omega_3 \omega_1 = 0$$

$$I_3 \ddot{\omega}_3 + (I_1 - I_1)\omega_1 \omega_2 = 0 \quad \implies \ddot{\omega}_3 = 0.$$ 

(b) Since $\ddot{\omega}_3 = 0$, then $\omega_3$ is constant and equal to its initial value, $\omega_3 = \omega_{30}$.

(c) Taking the derivative of the first Euler equation we get

$$I_1 \ddot{\omega}_1 + (I_3 - I_1)\omega_{30} \ddot{\omega}_2 = 0.$$ 

Substituting in for $\ddot{\omega}_2$ using the second equation we get

$$\ddot{\omega}_1 + \frac{(I_4 - I_1)^2 \omega_{30}^2}{I_1^2} \omega_1 = 0.$$
Defining $(I_3 - I_1)\omega_{30}/I_1 = \Omega$, we have

$$\ddot{\omega}_1 + \Omega^2 \omega_1 = 0.$$  

The general solution for $\omega_1$ is

$$\omega_1 = A \cos \Omega t + B \sin \Omega t$$

so that

$$\dot{\omega}_1 = \Omega [-A \sin \Omega t + B \cos \Omega t].$$

The initial conditions are $\omega_1(0) = \omega_{10}$ and (from the first Euler equation) $I_1 \dot{\omega}_1(0) = \tau_0 - (I_3 - I_1)\omega_3(0)\omega_2(0)$, or $\dot{\omega}_1(0) = \tau_0/I_1$. Applying these initial conditions to the general solution we get

$$\omega_{10} = \omega_1(0) = A, \quad \frac{\tau_0}{I_1} = \dot{\omega}_1(0) = B\Omega$$

so $A = \omega_{10}$ and $B = \tau_0/(I_1\Omega)$ and

$$\omega_1(t) = \omega_{10} \cos \Omega t + \frac{\tau_0}{I_1\Omega} \sin \Omega t.$$  

(d) The second Euler equation can be written $\dot{\omega}_2 = (I_3 - I_1)\omega_30\omega_1/I_1 = \Omega \omega_1$. Then to find $\omega_2(t)$ we can just integrate:

$$\omega_2(t) = \int_0^t \dot{\omega}_2 \, dt = \int_0^t \left[ \Omega \omega_{10} \cos \Omega t + \frac{\tau_0}{I_1} \sin \Omega t \right] dt$$

$$= \omega_{10} \sin \Omega t + \frac{\tau_0}{I_1\Omega} (1 - \cos \Omega t).$$

3.(a) The kinetic energy is

$$T = \frac{1}{2}m \dot{x}_1^2 + \frac{1}{2}m \dot{x}_2^2$$

so the $M$ matrix is

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}. $$

The $ij$ element of the $K$ matrix comes from the second derivative of the potential, $K_{ij} = \partial^2 V/\partial x_i \partial x_j$, so

$$K = \begin{pmatrix} \frac{\partial^2 V}{\partial x_1^2} & \frac{\partial^2 V}{\partial x_1 \partial x_2} \\ \frac{\partial^2 V}{\partial x_2 \partial x_1} & \frac{\partial^2 V}{\partial x_2^2} \end{pmatrix} = k \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}. $$

Since $M$ is already diagonal, we just rescale to find the $W$ matrix

$$W = M^{-1/2}K M^{-1/2} = \frac{1}{\sqrt{m}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} k \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix} \frac{1}{\sqrt{m}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{k}{m} \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix}. $$

The normal mode frequencies are given by the eigenvalues of $W$:

$$0 = \begin{vmatrix} 7 - \lambda & 2 \\ 2 & 4 - \lambda \end{vmatrix} = 24 - 11\lambda + \lambda^2 = (\lambda - 8)(\lambda - 3).$$
Therefore $\omega_1 = \sqrt{8k/m}$ and $\omega_2 = \sqrt{3k/m}$.

(b) In order to find the normal mode coordinates, we must find the eigenvectors corresponding to the eigenvalues from above. For the first eigenvalue $\lambda_1 = 8$:

$$0 = \begin{pmatrix} 7 - \lambda_1 & 2 \\ 2 & 4 - \lambda_1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 2C_2 - C_1 \\ 2C_1 - 4C_2 \end{pmatrix}$$

which implies $C_1 = 2C_2$ and the first eigenvector is $C_2(2, 1)$, or, after normalization, $(2, 1)/\sqrt{5}$. For the second eigenvalue $\lambda_2 = 3$:

$$0 = \begin{pmatrix} 7 - \lambda_2 & 2 \\ 2 & 4 - \lambda_2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 4C_1 + 2C_2 \\ 2C_1 + C_2 \end{pmatrix}$$

which implies $C_2 = -2C_1$ and the second eigenvector is $C_1(1, -2)$, or, after normalization, $(1, -2)/\sqrt{5}$. The transformation matrix $A$ is then $A = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$.

The normal mode coordinates are given by $q = AM^{1/2}x$, or

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \sqrt{m} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \sqrt{m/5} \begin{pmatrix} 2x_1 + x_2 \\ x_1 - 2x_2 \end{pmatrix}$$

so $q_1 = \sqrt{m/5}(2x_1 + x_2)$ and $q_2 = \sqrt{m/5}(x_1 - 2x_2)$.

(c) Since the off-diagonal elements of $W$ are not small with respect to the difference of the diagonal elements, we would not expect perturbation theory to give an answer that is close to the exact answer.