1. (a) Since the velocity doesn't depend explicitly on time, the flow is steady. The flow is incompressible if $\nabla \cdot \mathbf{v} = 0$:
\[
\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}
\]
\[
= (-2ay) + (2ay - 2by) + (2by) = 0
\]
So this is incompressible. The flow is irrotational if $\nabla \times \mathbf{v} = 0$:
\[
\nabla \times \mathbf{v} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-2ay & (a - b)y^2 & 2by
\end{vmatrix}
\]
\[
= \hat{x}(2bz - 0) + \hat{y}(0 - 0) + \hat{z}(0 - (-2ax)) = 2bz\hat{x} + 2ax\hat{z}.
\]
Hence the flow is rotational and has angular velocity $\omega = \nabla \times \mathbf{v}/2 = bz\hat{x} + ax\hat{z}$.

(b) Since the velocity does depend explicitly on time, the flow is not steady. Next we check
\[
\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}
\]
\[
= (2ay) + (2tz) + (2by) \neq 0
\]
So this fluid is compressible. Finally
\[
\nabla \times \mathbf{v} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
2axy + bz^2 & ax^2 + 2tzy & 2bxz + ty^2
\end{vmatrix}
\]
\[
= \hat{x}(2ty - 2ty) + \hat{y}(2bz - 2bz) + \hat{z}(2ax - 2ax) = 0
\]
so the flow is irrotational. The velocity potential function is
\[
\phi = \int_{(0,0,0)}^{(x,y,z)} \mathbf{v} \cdot d\mathbf{r} = \int_{(0,0,0)}^{(x,y,z)} (v_x\,dx + v_y\,dy + v_z\,dz)
\]
\[
= \left( \int_0^x (2axy + bz^2)\,dx \right) \bigg|_{y=z=0} + \left( \int_0^y (ax^2 + 2tzy)\,dy \right) \bigg|_{x=x,z=0} + \left( \int_0^z (2bxz + ty^2)\,dz \right) \bigg|_{x=x,y=y}
\]
\[
= 0 + (ax^2y) + (bxz^2 + ty^2z) = ax^2y + bxz^2 + ty^2z.
\]
2. (a) When in equilibrium the board should have its center of mass directly under the support point (since then gravity exerts no torque), as indicated in the diagram below:

![Diagram of a board with its center of mass and support point indicated.]

(b) We can use the fact that the moment of inertia is $ML^2/12$ for a long, thin rod about an axis perpendicular to the rod going through the center of the rod. The board has the same mass distribution as a thin rod (relative to an axis that bisects it), so it also has $I = ML^2/12$ about an axis in the plane of the board that goes through its center. If the $x$ axis bisects the board along the long direction, then $I_{x,CM} = Ma^2/12$. Similarly, if the $y$ axis bisects the board along its short direction, $I_{y,CM} = Mb^2/12$. Using the perpendicular axis theorem, the moment of inertia about an axis perpendicular to the board and through its center is $I_{z,CM} = I_{x,CM} + I_{y,CM} = M(a^2 + b^2)/12$. Now using the parallel axis theorem the moment of inertia for rotation about a point on a corner is $I = I_{z,CM} + MD^2$, where $D$ is the distance from the CM to the corner. In this case, $D^2 = (a^2 + b^2)/4$ (from the Pythagorean theorem), so $I = M(a^2 + b^2)/12 + M(a^2 + b^2)/4 = M(a^2 + b^2)/3$.

Another method to find $I$ would be by direct integration. Letting one corner be the origin and $\rho$ be the mass per unit area (constant):

$$I = \int_0^b \int_0^a \rho r^2 \, dy \, dx = \int_0^b \int_0^a \rho (x^2 + y^2) \, dy \, dx$$
$$= \rho \int_0^b \left( \frac{x^3}{3} + \frac{y^3}{3} \right) \bigg|_0^a = \rho \int_0^b \left( \frac{1}{3} a^3 + ax^2 \right) \, dx$$
$$= \rho \left( \frac{1}{3} a^3 x + \frac{1}{3} ax^3 \right) \bigg|_0^b = \frac{1}{3} \rho (a^3 b + ab^3) = \frac{1}{3} \rho ab(a^2 + b^2).$$

Since the mass is just $M = \rho ab$, then $I = M(a^2 + b^2)/3$.

(c) The angular frequency of small oscillations is just $\omega = \sqrt{gh/k_O}$, where $h$ is the distance from the CM and the point about which the rotation takes place and $k_O$ is the radius of gyration about that point. In this case $h = \sqrt{a^2 + b^2}/2$ and $k_O^2 = I/M = (a^2 + b^2)/3$ (using the results for $I$ from part (b) above). Therefore

$$\omega = \sqrt{\frac{g\sqrt{a^2 + b^2}}{2} \frac{3}{(a^2 + b^2)}} = \sqrt{\frac{3g}{2\sqrt{a^2 + b^2}}}.$$
(d) The distance of the center of oscillation from the support point is

$$\ell = \frac{k^2 O}{h} = \frac{a^2 + b^2}{3} \frac{2}{\sqrt{a^2 + b^2}} = \frac{2}{3} \sqrt{a^2 + b^2}.$$

The center of oscillation lies along a line that goes from one corner through the CM to the opposite corner, about two-thirds of the way from the support point, or one-third of the way from the opposite corner (see point O' on the diagram).

3. Using the usual axes $\hat{x}$ to the right and $\hat{y}$ upward on the page, the net force is $3 \, \hat{x} - 4 \, \hat{y}$. The net torque (taking torques about the CM) is $\tau = (3 \, \text{N})(1 \, \text{m}) + (6 \, \text{N})(2 \, \text{m}) - (10 \, \text{N})(1 \, \text{m}) = 5 \, \text{N}\cdot\text{m}$, where positive is counterclockwise. The equilibrant must therefore have the opposite force and torque, i.e., $-3 \, \hat{x} + 4 \, \hat{y}$ and $\tau = -5 \, \text{N}\cdot\text{m}$ (clockwise). The magnitude of the equilibrant is 5 N, so the perpendicular distance from the line of action to the CM must be $r_\perp = \tau / F = 5 \, \text{N}\cdot\text{m} / 5 \, \text{N} = 1 \, \text{m}$. The equilibrant makes an angle of $\arctan(3/4) = 53^\circ$ with respect to the $-x$ axis, so the perpendicular distance from the line of action of the force and the CM makes an angle 37° with respect to the $x$ axis (see diagram). The force can act at the point on the line of action closest to the CM (i.e., 1 m away), or anywhere along that line.

4. (a) This is similar to the homework problem #3 on Set #4, about a bending beam, except in this case the ends are free and so there is no torque applied at the ends. Let $F_0$ be the support force at the left end and $F_1$ be the support force at the right end. For the shear force $S(x)$ acting on the board at point $x$,

$$S = -\sum_{x_i < x} F_i + \int_0^x \frac{Mg}{L} dx' = -F_0 + \frac{Mg}{L} x.$$

$F_1$ comes in as the force at $x = L$:

$$F_1 = S(L) = -F_0 + \frac{Mg}{L} L = -F_0 + M g \quad \Rightarrow \quad F_0 + F_1 = M g$$

which makes sense since the two support forces must cancel the weight of the board. From the symmetry of the problem, we can then say $F_0 = F_1 = M g/2$. This is the same as the homework problem. In fact,
from the symmetry, and the fact that there are just two support forces, we could infer this without a real
calculation.

For the torque acting at $x$ we have

$$\tau = F_0 x - \int_0^x \frac{Mg}{L} (x - x') dx' = \frac{Mg}{2} x - \frac{1}{2} \frac{Mg}{L} x^2.$$  

where we have used $F_0 = Mg/2$. As a check, the torque at $x = L$ should be zero:

$$\tau(L) = \frac{Mg}{2} L - \frac{1}{2} \frac{Mg}{L} L^2 = 0 \quad \text{(checks)}.$$  

Now we can use the basic equation that relates the shape of the beam to the torque (ignoring the shear term):

$$\frac{d^2 y}{dx^2} = \frac{\tau}{Yk^2 A} = \frac{1}{Yk^2 A} \left( \frac{Mg}{2} x - \frac{1}{2} \frac{Mg}{L} x^2 \right).$$  

Integrating once

$$\frac{dy}{dx} = \left( \frac{dy}{dx} \right)_{x=0} + \frac{1}{Yk^2 A} \left( \frac{1}{4} \frac{Mg}{L} x^2 - \frac{1}{6} \frac{Mg}{L} x^3 \right)$$  

where the constant of integration is not zero in this case since the slope at the left end ($x = 0$) is not fixed (the board is free, other than the support force). From the symmetry, the slope at the right end should be a mirror reflection of the left end, i.e., the slope at the right end is the negative of the slope at the left end, so

$$- \left( \frac{dy}{dx} \right)_{x=0} = \left( \frac{dy}{dx} \right)_{x=L} = \left( \frac{dy}{dx} \right)_{x=0} + \frac{1}{Yk^2 A} \left( \frac{1}{4} \frac{Mg}{L} x^2 - \frac{1}{6} \frac{Mg}{L} x^3 \right) + \frac{MgL^2}{12Yk^2 A}.$$  

Therefore the slope at the left end is $-\frac{MgL^2}{(24Yk^2 A)}$. We could also have used the symmetry to say that the slope of the board in the middle, $y = L/2$, was zero; this also would have given us the slope at the left end.

Next we can integrate the slope to get

$$y = \frac{1}{Yk^2 A} \left( -\frac{1}{24} MgL^2 x + \frac{1}{12} Mgx^3 - \frac{1}{24} \frac{Mg}{L} x^4 \right) = \frac{Mgx}{24Yk^2 AL} (-L^3 + 2x^2 L - x^3).$$  

where the constant of integration is zero since the left end of the board is at $y = 0$. It is not hard to check that the right end of the board is also at $y = 0$, i.e., $y(L) = 0$, as it should be.

(b) The position of the board at the middle is $y(L/2) = -\frac{5MgL^3}{(384Yk^2 A)}$. This is 5 times lower than when the board is clamped at the end (forced to have zero slope there). We found the slope of the board at the left end ($y = 0$) in part (a); it was $-\frac{MgL^2}{(24Yk^2 A)}$. The slope at the right end is the negative of this, $\frac{MgL^2}{(24Yk^2 A)}$. Note that by measuring the weight, length, cross section and position of the middle of the middle of the board, we could determine the Young’s modulus of the material.