1. (a) The kinetic energy is simply

\[ T = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 (x_1^2 + y_1^2) + \frac{1}{2} m_2 (x_2^2 + y_2^2). \]

The potential energy has a term for each spring

\[ V = \frac{1}{2} k (x_1^2 + y_1^2) + \frac{1}{2} k (x_2^2 + y_2^2) + \frac{1}{2} k [(x_1 - x_2)^2 + (y_1 - y_2)^2] \]

where the first term comes from \( m_1 \) being attached to the origin, the second from \( m_2 \) being attached to the origin, and the third from \( m_1 \) being attached to \( m_2 \). So

\[ L = T - V = \frac{1}{2} m_1 (x_1^2 + y_1^2) + \frac{1}{2} m_1 (x_2^2 + y_2^2) - \frac{1}{2} k (x_1^2 + y_1^2) - \frac{1}{2} k (x_2^2 + y_2^2) - \frac{1}{2} k [(x_1 - x_2)^2 + (y_1 - y_2)^2]. \]

(b) The Lagrange equation of motion for \( x_1 \) is

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_1} \right) = \frac{\partial L}{\partial x_1} \]

\[ \frac{d}{dt} (m_1 \dot{x}_1) = -kx_1 - k(x_1 - x_2) \]

\[ m_1 \ddot{x}_1 = -kx_1 - k(x_1 - x_2). \]

The Lagrange equation of motion for \( y_1 \) is

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}_1} \right) = \frac{\partial L}{\partial y_1} \]

\[ \frac{d}{dt} (m_1 \dot{y}_1) = -ky_1 - k(y_1 - y_2) \]

\[ m_1 \ddot{y}_1 = -ky_1 - k(y_1 - y_2). \]

The Lagrange equation of motion for \( x_2 \) is

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}_2} \right) = \frac{\partial L}{\partial x_2} \]

\[ \frac{d}{dt} (m_2 \dot{x}_2) = -kx_2 - k(x_2 - x_1) \]

\[ m_2 \ddot{x}_2 = -kx_2 - k(x_2 - x_1). \]

The Lagrange equation of motion for \( y_2 \) is

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}_2} \right) = \frac{\partial L}{\partial y_2} \]

\[ \frac{d}{dt} (m_2 \dot{y}_2) = -ky_2 - k(y_2 - y_1) \]

\[ m_2 \ddot{y}_2 = -ky_2 - k(y_2 - y_1). \]
We note that \( x_1 \) and \( x_2 \) are coupled together. Assuming solutions of the form \( x_j = C_j e^{pt} \), then \( \ddot{x}_j = p^2 C_j e^{pt} \) and
\[
C_1(m_1p^2 + 2k) - C_2 k = 0
\]
\[
-C_1 k + C_2(m_2p^2 + 2k) = 0.
\]
The characteristic equation for this system is
\[
\begin{vmatrix}
  m_1p^2 + 2k & -k \\
  -k & m_2p^2 + 2k
\end{vmatrix}
= m_1m_2p^4 + 2k(m_1 + m_2)p^2 + 3k^2 = 0.
\]
The solutions to this (from the quadratic formula, solving for \( p^2 \)) are
\[
p^2 = \frac{-2k(m_1 + m_2) \pm \sqrt{4(m_1 + m_2)^2k^2 - 12k^2m_1m_2}}{2m_1m_2}.
\]
These may be simplified to
\[
p^2 = \frac{-k(m_1 + m_2) \pm k\sqrt{(m_1 + m_2)^2 - 3m_1m_2}}{m_1m_2}.
\]
After defining \( \omega_1^2 = -p_1^2 \) and \( \omega_2^2 = -p_2^2 \) we get the two normal mode solutions
\[
\omega_1 = \sqrt{\frac{k(m_1 + m_2) + k\sqrt{(m_1 + m_2)^2 - 3m_1m_2}}{m_1m_2}}, \quad \omega_2 = \sqrt{\frac{k(m_1 + m_2) - k\sqrt{(m_1 + m_2)^2 - 3m_1m_2}}{m_1m_2}}.
\]
The solutions for \( p^2 \) are real since the quantity in the square root is positive; this can be seen by expanding it out to get \( (m_1 + m_2)^2 - 3m_1m_2 = m_1^2 + m_2^2 - m_1m_2 = (m_1 - m_2)^2 + m_1m_2 > 0 \). The solutions for \( p^2 \) are negative since \( p^2 \) is a positive quantity plus or minus something that is smaller (the square root is smaller than the first term since it is the square root of the first term minus a positive quantity). Therefore the actual roots are
\[
p = \pm i\omega_1, \quad \pm i\omega_2
\]
which means that the solutions are all oscillatory (no real part to the root means no exponential damping). The equations for \( y_1 \) and \( y_2 \) lead to identical results for the normal mode angular frequencies.

2. (a) The position of \( m_1 \) as measured from the origin is
\[
x_1 = L_1 \sin \theta_1, \quad y_1 = -L_1 \cos \theta_1
\]
and the position of \( m_2 \) as measured from the origin is
\[
x_2 = L_1 \sin \theta_1 + L_2 \sin \theta_2, \quad y_2 = -L_1 \cos \theta_1 - L_2 \cos \theta_2.
\]
The velocities are therefore
\[
\dot{x}_1 = L_1 \cos \theta_1 \dot{\theta}_1, \quad \dot{y}_1 = L_1 \sin \theta_1 \dot{\theta}_1
\]
\[
\dot{x}_2 = L_1 \cos \theta_1 \dot{\theta}_2 + L_2 \cos \theta_2 \dot{\theta}_2, \quad \dot{y}_2 = L_1 \sin \theta_1 \dot{\theta}_1 + L_2 \sin \theta_2 \dot{\theta}_2.
\]
Therefore the kinetic energy is

\[ T = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} m_1 (\dot{x}_2^2 + \dot{y}_2^2) \]

\[ = \frac{1}{2} m_1 (L_1^2 \cos^2 \theta_1 \dot{\theta}_1^2 + L_1^2 \sin^2 \theta_1 \dot{\theta}_1^2) + \frac{1}{2} m_2 (L_2^2 \cos^2 \theta_2 \dot{\theta}_2^2 + L_2^2 \sin^2 \theta_2 \dot{\theta}_2^2 + 2L_1L_2 \cos \theta_1 \cos \theta_2 \dot{\theta}_1 \dot{\theta}_2) \]

\[ + \frac{1}{2} m_2 (L_2^2 \cos^2 \theta_2 \dot{\theta}_2^2 + L_2^2 \sin^2 \theta_2 \dot{\theta}_2^2 + 2L_1L_2 \sin \theta_2 \dot{\theta}_1 \dot{\theta}_2) \]

\[ = \frac{1}{2} m_1 L_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 [L_2^2 \dot{\theta}_2^2 + L_2^2 \dot{\theta}_2^2 + 2L_1L_2 \cos(\theta_1 - \theta_2) \dot{\theta}_1 \dot{\theta}_2]. \]

The potential energy is

\[ V = m_1 g y_1 + m_2 g y_2 = -(m_1 + m_2) g L_1 \cos \theta_1 - m_2 g L_2 \cos \theta_2. \]

The Lagrangian is therefore

\[ L = T - V = T + (m_1 + m_2) g L_1 \cos \theta_1 + m_2 g L_2 \cos \theta_2. \]

(b) The Lagrange equation of motion for \( \theta_1 \) is

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = 0 \]

\[ \frac{d}{dt} \left[ (m_1 + m_2) L_1^2 \dot{\theta}_1 + m_2 L_1 L_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right] - \left[ -(m_1 + m_2) g L_1 \sin \theta_1 - m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \right] = 0 \]

\[ (m_1 + m_2) L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) - m_2 L_1 L_2 \dot{\theta}_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2) \]

\[ + (m_1 + m_2) g L_1 \sin \theta_1 + m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) = 0 \]

\[ (m_1 + m_2) L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + m_2 L_1 L_2 \dot{\theta}_2 \dot{\theta}_1 \sin(\theta_1 - \theta_2) + (m_1 + m_2) g L_1 \sin \theta_1 = 0 \]

The Lagrange equation of motion for \( \theta_2 \) is

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = 0 \]

\[ \frac{d}{dt} \left[ m_2 L_2^2 \dot{\theta}_2 + m_2 L_1 L_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \right] - \left[ -m_2 g L_2 \sin \theta_2 - m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \right] = 0 \]

\[ m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) + m_2 L_1 L_2 \dot{\theta}_1 (\dot{\theta}_2 - \dot{\theta}_1) \sin(\theta_1 - \theta_2) \]

\[ + m_2 g L_2 \sin \theta_2 - m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) = 0 \]

\[ m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g L_2 \sin \theta_2 = 0 \]

In the limit that \( \theta_1 \) and \( \theta_2 \) are small, the cosines go to unity and the sines are linear in the argument (e.g., \( \sin \theta_1 \approx \theta_1 \)). Furthermore, the \( \dot{\theta}_1 \sin(\theta_1 - \theta_2) \) and \( \dot{\theta}_2 \sin(\theta_1 - \theta_2) \) terms have three powers of an angle and can be ignored. Also, \( m_1 = m_2 = m \) and \( L_1 = L_2 = L \). Therefore the equations of motion reduce to

\[ 2m L_2^2 \ddot{\theta}_1 + m L_2^2 \ddot{\theta}_2 + 2mgL \dot{\theta}_1 = 0 \]

\[ mL^2 \ddot{\theta}_2 + mL^2 \ddot{\theta}_1 + mgL \dot{\theta}_2 = 0. \]
This has the form of coupled harmonic oscillators, so we assume solutions \( \theta_1 = C_1 e^{pt} \) and \( \theta_2 = C_2 e^{pt} \) and get the following equations

\[
\begin{align*}
2m(L^2p^2 + gL)C_1 + mL^2p^2C_2 &= 0 \\
mL^2p^2C_1 + m(L^2p^2 + gL)C_2 &= 0
\end{align*}
\]

which has characteristic equation

\[
\begin{vmatrix}
2m(L^2p^2 + gL) & mL^2p^2 \\
mL^2p^2 & m(L^2p^2 + gL)
\end{vmatrix} = m^2L^4p^4 + 4m^2gL^3p^2 + 2m^2g^2L^2 = 0
\]

\[
p^4 + 4\frac{g}{L}p^2 + 2\frac{g^2}{L^2} = 0
\]

where the last line comes from dividing by \( m^2L^4 \). This has solution

\[
p^2 = \frac{1}{2} \left( -\frac{4g}{L} \pm \sqrt{\left(\frac{4g}{L}\right)^2 - 8\frac{g^2}{L^2}} \right) = -\frac{g}{L}(2 \pm \sqrt{2}).
\]

These solutions clearly have negative \( p^2 \), so the normal modes are pure oscillations with no damping. The normal mode angular frequencies are

\[
\omega_1, \omega_2 = \sqrt{-p^2} = \sqrt{\frac{g}{L}(2 \pm \sqrt{2})}.
\]