Physics 361 - HOMEWORK SET #8
Solutions

1. (a) Using Newton’s 2nd law we find

\[ m_1 \ddot{x}_1 = -k(x_1 + L - x_2) \]
\[ m_2 \ddot{x}_2 = k(x_1 + L - x_2). \]

Note that the force on \( m_1 \) is to the left (away from \( x_2 \)) if \( x_1 + L > x_2 \), i.e., if the spring is compressed to be shorter than \( L \), which makes sense since the masses would repel each other in this case. Likewise, the force on \( m_1 \) is to the right (towards \( x_2 \)) if \( x_1 + L < x_2 \), i.e., if the spring is stretched to be longer than \( L \), which makes sense since the masses would be pulled together in this case. Of course the force on \( m_2 \) is equal and opposite to the force on \( m_1 \) (Newton’s 3rd law). The CM is defined by

\[ M X = m_1 x_1 + m_2 x_2, \]

where \( M = m_1 + m_2 \) is the total mass. Therefore

\[ M \dot{X} = m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = 0. \]

Clearly the CM position \( X \) has no acceleration, and hence moves at constant velocity.

The relative coordinate is \( x = x_1 - x_2 \), so \( \ddot{x} = \ddot{x}_1 - \ddot{x}_2 \). Dividing each equation by the appropriate mass and subtracting we get

\[ \ddot{x} = \ddot{x}_1 - \ddot{x}_2 = - \left[ \frac{k}{m_1} + \frac{k}{m_2} \right] (x_1 + L - x_2) \]
\[ \ddot{x} = - \frac{k(m_1 + m_2)}{m_1 m_2} (x + L) \]
\[ \ddot{x} + \frac{k}{\mu} x = - \frac{k}{\mu} L \]

where we have used the fact that the reduced mass is \( \mu = m_1 m_2 / (m_1 + m_2) \). Clearly the coordinate \( x \) has simple harmonic motion with \( \omega^2 = k/\mu \), or \( \omega = \sqrt{k/\mu} \). The \( k/\mu \) term just tells us that the particular solution for \( x \) is \( x = -L \).

(b) From the equations of motion we can immediately write down the solutions

\[ X = X_0 + V_0 t \]
\[ x = A \cos \omega t + B \sin \omega t - L \]

where the last term in \( x \) is the particular solution. Therefore

\[ V = V_0 \]
\[ v = -\omega A \sin \omega t + \omega B \cos \omega t. \]

From the initial conditions we have \( X_0 = X(0) = \frac{m_1(0) + m_2(L)}{M} = m_2 L/M \), \( V_0 = V(0) = \frac{m_1(v_0) + m_2(0)}{M} = m_1 v_0/M \), \( x(0) = 0 - L = -L \) and \( v(0) = v_0 - 0 \), so

\[ -L = x(0) = A \cos 0 + B \sin 0 - L = A - L \]
\[ v_0 = v(0) = -\omega A \sin 0 + \omega B \cos 0 = \omega B \]
and so \( A = 0, \ B = v_0/\omega \). Therefore
\[
X = \frac{m_2L}{M} + \frac{m_1v_0}{M}t \\
x = (v_0/\omega) \sin \omega t - L.
\]
Inverting the equations for \( X \) and \( x \) we can solve for \( x_1 \) and \( x_2 \) to get
\[
x_1 = X + \frac{m_2}{M}x, \quad x_2 = X - \frac{m_1}{M}x
\]
which, after substituting in the solutions for \( X \) and \( x \), gives
\[
x_1 = \frac{m_2L}{M} + \frac{m_1v_0}{M}t + \frac{m_2}{M} (\frac{v_0}{\omega} \sin \omega t - L) = \frac{m_1v_0}{M}t + \frac{m_2v_0}{M\omega} \sin \omega t \\
x_2 = \frac{m_2L}{M} + \frac{m_1v_0}{M}t - \frac{m_1}{M} (\frac{v_0}{\omega} \sin \omega t - L) = \frac{m_1v_0}{M}t - \frac{m_1v_0}{M\omega} \sin \omega t.
\]

2. (a) The equations of motion are
\[
m\ddot{x}_1 = -kx_1 - k(x_1 - x_2) \\
m\ddot{x}_2 = k(x_1 - x_2).
\]
These will have general solution \( x_1 = C_1 e^{pt} \) and \( x_2 = C_2 e^{pt} \). Plugging into the equations of motion we get
\[
C_1(mp^2 + 2k) - C_2k = 0 \\
-C_1k + C_2(mp^2 + k) = 0
\]
This can only have non-trivial solutions for \( C_1 \) and \( C_2 \) if the determinant of the coefficient matrix is zero, i.e.,
\[
\begin{vmatrix}
mp^2 + 2k & -k \\
-k & mp^2 + k
\end{vmatrix} = 0 \quad \implies \quad m^2p^4 + 3mp^2 + k^2 = 0.
\]
This has two solutions via the quadratic equation: \( p^2 = (-3\pm\sqrt{5})\omega_0^2/2 \), where \( \omega_0^2 = k/m \). Both solutions for \( p^2 \) are negative, so both solutions lead to oscillations. Defining \( \omega_1^2 = (3 - \sqrt{5})\omega_0^2/2 \) and \( \omega_2^2 = (3 + \sqrt{5})\omega_0^2/2 \) (which are the negatives of the two roots for \( p^2 \)), the general solution for \( x_1 \) can be written
\[
x_1 = A_1 \cos \omega_1 t + B_1 \sin \omega_1 t + A_2 \cos \omega_2 t + B_2 \sin \omega_1 t \\
x_2 = A'_1 \cos \omega_1 t + B'_1 \sin \omega_1 t + A'_2 \cos \omega_2 t + B'_2 \sin \omega_1 t
\]
We must determine the relationship of the \( x_2 \) coefficients using the relations above between the \( x_2 \) coefficients and the \( x_1 \) coefficients; from above we have \( C_2 = C_1(mp^2 + 2k)/k = C_1(p^2 + 2\omega_0^2)/\omega_0^2 \), where \( C_2 \) is an \( x_2 \) coefficient and \( C_1 \) is an \( x_1 \) coefficient. For \( p^2 = -\omega_1^2 \) we get \( C_2 = C_1(-\omega_1^2 + 2\omega_0^2)/\omega_0^2 = (1 + \sqrt{5})/2 > 0 \). Therefore \( A'_1 = (1 + \sqrt{5})A_1/2 \) and \( B'_1 = (1 + \sqrt{5})B_1/2 \). In this normal mode, the objects move in phase with each other (usually the case for the slower oscillation). For \( p^2 = -\omega_2^2 \) we get \( C_2 = C_1(-\omega_2^2 + 2\omega_0^2)/\omega_0^2 = (1 - \sqrt{5})/2 < 0 \). Therefore \( A'_2 = (1 - \sqrt{5})A_2/2 \) and \( B'_2 = (1 - \sqrt{5})B_2/2 \). In this
normal mode the objects move out of phase with each other. The complete solution is
\[ x_1 = A_1 \cos \omega_1 t + B_1 \sin \omega_1 t + A_2 \cos \omega_2 t + B_2 \sin \omega_1 t \]
\[ x_2 = \frac{1 + \sqrt{5}}{2} [A_1 \cos \omega_1 t + B_1 \sin \omega_1 t] + \frac{1 - \sqrt{5}}{2} [A_2 \cos \omega_2 t + B_2 \sin \omega_2 t]. \]

The velocities are
\[ v_1 = \omega_1 (-A_1 \sin \omega_1 t + B_1 \cos \omega_1 t) + \omega_2 (-A_2 \sin \omega_2 t + B_2 \cos \omega_2 t) \]
\[ v_2 = \frac{1 + \sqrt{5}}{2} \omega_1 [-A_1 \sin \omega_1 t + B_1 \cos \omega_1 t] + \frac{1 - \sqrt{5}}{2} \omega_2 [-A_2 \sin \omega_2 t + B_2 \cos \omega_2 t]. \]

(b) Applying the initial conditions
\[ 0 = x_1(0) = A_1 + A_2 \quad \implies \quad A_2 = -A_1 \]
\[ A = x_2(0) = \frac{1 + \sqrt{5}}{2} A_1 + \frac{1 - \sqrt{5}}{2} A_2 = \left[ \frac{1 + \sqrt{5}}{2} - \frac{1 - \sqrt{5}}{2} \right] A_1 = \sqrt{5} A_1 \quad \implies \quad A_1 = \frac{A}{\sqrt{5}} \]
\[ 0 = v_1(0) = B_1 \omega_1 + B_2 \omega_2 \]
\[ 0 = v_2(0) = \frac{1 + \sqrt{5}}{2} \omega_1 B_1 + \frac{1 - \sqrt{5}}{2} \omega_2 B_2. \]

The last two equations give \( B_1 = B_2 = 0 \), so the solution is
\[ x_1 = \frac{A}{\sqrt{5}} (\cos \omega_1 t - \cos \omega_2 t) \]
\[ x_2 = \frac{A}{2\sqrt{5}} \left[ (1 + \sqrt{5}) \cos \omega_1 t - (1 - \sqrt{5}) \cos \omega_2 t \right]. \]

3. (a) The equations of motion are the same as in Problem 2 except there is now an additional force acting on the second object:
\[ m \ddot{x}_1 = -k x_1 - k (x_1 - x_2) \]
\[ m \ddot{x}_2 = k (x_1 - x_2) + F_0 \cos \omega t. \]

We assume a form for the motion that is similar to the driving force, i.e.,
\[ x_1 = A_1 \cos \omega t, \quad x_2 = A_2 \cos \omega t. \]

After plugging into the equations of motion and combining terms we find (factors of \( \cos \omega t \) cancel everywhere)
\[ A_1 (-m \omega^2 + 2k) - A_2 k = 0 \quad \implies \quad A_1 (-\omega^2 + 2\omega_0^2) - A_2 \omega_0^2 = 0 \]
\[ -A_1 k + A_2 (-m \omega^2 + k) = F_0 \quad \implies \quad -A_1 + A_2 (-\omega^2 + \omega_0^2) = F_0 / m \]

where the right equations were obtained by dividing everything by \( m \) and using \( k/m = \omega_0^2 \). Note that no \( \sin \omega t \) terms were needed since there were no first derivatives.
(such as might come from drag forces). These are just two linear equations with two unknowns, which may be solved to yield

\[ A_1 = \frac{F_0}{m} \frac{\omega_0^2}{\omega^4 - 3\omega^2\omega_0^2 + \omega_0^4}, \quad A_2 = \frac{F_0}{m} \frac{2\omega_0^2 - \omega^2}{\omega^4 - 3\omega^2\omega_0^2 + \omega_0^4}. \]

(b) The solutions blow up (i.e., \(x_1\) and \(x_2\) go to infinity) when the denominator \(\omega^4 - 3\omega^2\omega_0^2 + \omega_0^4\) is zero. This occurs when (using the quadratic formula) \(\omega^2 = (3 \pm \sqrt{5})\omega_0^2/2\). Of course, these are just the two normal mode angular frequencies found in Problem 2.