1. First we find the velocity

\[ r = (x_0 + at^2, bt^3, ct) \]
\[ v = \frac{dr}{dt} = (2at, 3bt^2, c) \]

Then

\[ L = m r \times v = m \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ x_0 + at^2 & bt^3 & ct \\ 2at & 3bt^2 & c \end{vmatrix} \]
\[ = m [\dot{x}(cbt^3 - 3bt^2ct) - \dot{y}(c(x_0 + at^2) - 2atct) + \dot{z}(3bt^2(x_0 + at^2) - 2atbt^3)] \]
\[ = -2mbct^3\dot{x} - m(cx_0 - act^2)\dot{y} + m(3bx_0t^2 + abt^4)\dot{z}. \]

The force is

\[ F = m\ddot{r} = m\dot{v} = m(2a, 6bt, 0) = (2ma, 6mbt, 0). \]

The torque is

\[ \tau = \frac{dL}{dt} = [-6mbct^2\dot{x} + 2mact\dot{y} + m(6bx_0t + 4abt^3)\dot{z}] . \]

and also

\[ \tau = r \times F = \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ x_0 + at^2 & bt^3 & ct \\ 2ma & 6mbt & 0 \end{vmatrix} \]
\[ = \dot{x}(-6mbct) - \dot{y}(-2mabt^3) + \dot{z}(6mbt(x_0 + at^2) - 2mabt^3) \]
\[ = -6mbct^2\dot{x} + 2mact\dot{y} + (6mbx_0t + 4mabt^3)\dot{z} \]

which is the same as \( \frac{dL}{dt} \).

2. The vector equation is

\[ ma = -kr - bv \]

with component equations given by

\[ m\ddot{x} = -kx - b\dot{x}, \quad m\ddot{y} = -ky - b\dot{y}, \quad m\ddot{z} = -kz - b\dot{z}. \]

Each of these is a 1-D damped harmonic oscillator. Since it is underdamped the solutions look like

\[ x = (A_x \cos \omega_1 t + B_x \sin \omega_1 t)e^{-\gamma t} \]

with similar solutions for \( y \) and \( z \), and \( \omega_1^2 = \omega_0^2 - \gamma^2 \), \( \omega_0^2 = k/m \) and \( \gamma = b/(2m) \). The velocity is

\[ v_x = [(-\gamma B_x - A_x \omega_1) \sin \omega_1 t + (\omega_1 B_x - \gamma A_x) \cos \omega_1 t] e^{-\gamma t} \]
with similar solutions for \( v_y \) and \( v_z \). Using the initial condition \( \mathbf{r}_0 = (x_0, 0, 0) \) we find

\[
A_x = x_0, \quad A_y = 0, \quad A_z = 0
\]

and using \( \mathbf{v}_0 = (0, v_0, 0) \) we get

\[
B_x = \gamma x_0/\omega_1, \quad B_y = v_0/\omega_1, \quad B_z = 0
\]

so the solution is

\[
\mathbf{r}(t) = [x_0(\cos \omega_1 t + (\gamma/\omega_1) \sin \omega_1 t)e^{-\gamma t}, (v_0/\omega_1) \sin \omega_1 t e^{-\gamma t}, 0].
\]

After \( n \) cycles, \( t = nT \), where \( T = 2\pi/\omega_1 \) is the period. Then \( \cos \omega_1 t = \cos(2\pi n) = 1 \) and \( \sin \omega_1 t = \sin(2\pi n) = 0 \), and so \( x = x_0 e^{-\gamma nT} \) and \( y = 0 \). The distance from the origin after \( n \) cycles is then

\[
r = \sqrt{x^2 + y^2} = x_0 e^{-\gamma nT} = x_0 e^{-2\pi \gamma n/\omega_1}.
\]

3. A force is conservative if \( \nabla \times \mathbf{F} = 0 \).

(a) This is in Cartesian coordinates so we have

\[
\nabla \times \mathbf{F} = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\partial / \partial x & \partial / \partial y & \partial / \partial z \\
2axy^2 + by & 2ax^2 y + bx & ax^2 y^2 \\
\end{vmatrix}
= \hat{x}(2ax^2 y - 2axy^2) - \hat{y}(2axy^2 - 2axy^2) + \hat{z}(4axyz + b - 4axyz - b) = 0
\]

so this force is conservative. To find the potential energy \( V = -\int \mathbf{F} \cdot d\mathbf{r} \) we can integrate from the origin to a general position \((x, y, z)\) via the path \((0, 0, 0) \rightarrow (x, 0, 0) \rightarrow (x, y, 0) \rightarrow (x, y, z)\). This gives three terms

\[
V = -\left( \int_0^x F_x \, dx \right)_{y=z=0} - \left( \int_0^y F_y \, dy \right)_{x=x,z=0} - \left( \int_0^z F_z \, dz \right)_{x=x,y=y}.
\]

The integrals are straightforward and give

\[
V = -bxy - (ax^2 y^2 z) = -ax^2 y^2 z - bxy.
\]

Note that the first integral is zero since \( z = 0 \) along that part of the path, and one of the terms in the second integral is also zero for the same reason. Since it is a conservative force, taking any other path should give the same result.

(b) This force is in cylindrical coordinates so

\[
\nabla \times \mathbf{F} = \hat{\rho} \left[ \frac{1}{\rho} \frac{\partial F_\phi}{\partial \phi} - \frac{\partial F_\rho}{\partial z} \right] + \hat{\phi} \left[ \frac{\partial F_\rho}{\partial z} - \frac{\partial F_\phi}{\partial \rho} \right] + \hat{z} \left[ \frac{1}{\rho} \frac{\partial (\rho F_\phi)}{\partial \rho} - \frac{1}{\rho} \frac{\partial F_\rho}{\partial \phi} \right].
\]
In this case $F_\rho = -3a\rho^2 \cos \phi$, $F_\phi = a\rho^2 \sin \phi$ and $F_z = 3az^2$, so

$$\nabla \times F = \hat{\rho}[0 - 0] + \hat{\phi}[0 - 0] + \hat{z}\left[\frac{1}{\rho}(3a\rho^2 \sin \phi) - \frac{1}{\rho}(3a\rho^2 \sin \phi)\right] = 0.$$ 

Thus this force is also conservative. To find the potential, we can integrate in $(\rho, \phi, z)$ space from $(0, 0, 0) \rightarrow (\rho, \phi, 0) \rightarrow (\rho, \phi, z)$ where the first step has $\rho$ varying with $\phi$ set at a constant value and the second has $z$ varying with $\rho$ and $\phi$ fixed, i.e.,

$$V = -\left(\int_0^\rho F_\rho d\rho\right)_{\phi=\phi, z=0} - \left(\int_0^z F_z dz\right)_{\rho=\rho, \phi=\phi}.$$

The integrals are easy to do and

$$V = -(a\rho^3 \cos \phi) - (az^3) = a\rho^3 \cos \phi - az^3.$$

Note that we did not have to take three steps, since the first step was done in the radial direction at the angle $\phi$, which took care of both the radial distance and angle in the $x-y$ plane. This was possible since at $\rho = 0$, any value of $\phi$ is allowed. If we had done an integral with $\phi$ varying, it would have looked like $\int F_\rho \rho d\phi$, since $\rho d\phi$ is the infinitesimal displacement along the $\phi$ direction (it must have dimensions of distance).

(c) This force is in spherical polar coordinates so

$$\nabla \times F = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta}(F_\phi \sin \theta) - \frac{\partial F_\theta}{\partial \phi}\right] \hat{\rho} + \frac{1}{r} \left[\frac{\partial F_\rho}{\partial \theta} - \frac{\partial F_\theta}{\partial \rho}\right] \hat{\theta} + \frac{1}{r} \left[\frac{\partial F_\rho}{\partial r} - \frac{\partial F_\phi}{\partial \theta}\right] \hat{\phi}.$$

In this case $F_\rho = -2ar \sin \theta \cos \phi$, $F_\theta = -ar \cos \theta \cos \phi$ and $F_\phi = ar \sin \theta \sin \phi$. Looking at the $\hat{\rho}$ component we get $[(2ar \sin \theta \cos \phi \sin \phi) - (ar \cos \theta \sin \phi)] / (r \sin \theta) \neq 0$, so this force is not conservative. Note: if you calculated the $\hat{\theta}$ and $\hat{\phi}$ components of $\nabla \times \mathbf{F}$, they should be zero.

4. (a) Substituting $r = \sqrt{x^2 + y^2}$ and $\cos \theta = x/r$ into $r = r_0 \cos \theta$ gives $\sqrt{x^2 + y^2} = r_0 x / \sqrt{x^2 + y^2}$, or $x^2 + y^2 = r_0^2$. By completing the square in $x$ we get $(x - r_0/2)^2 + y^2 = r_0^2/4$. This is a circle with radius $\sqrt{r_0^2/4} = r_0/2$ and centered at $(r_0/2, 0)$ in the $x-y$ plane.

(b) Converting from $r$ to the variable $u = 1/r$, we get $u = 1/(r_0 \cos \theta)$ and $F = k/u^n$. Then Eq. 3.222 in the text is $d^2u/d\theta^2 = -u - (m/L^2u^2)(k/u^n)$. Finding the derivatives of $u$:

$$\frac{du}{d\theta} = \frac{1}{r_0 \cos \theta} \sin \theta, \quad \frac{d^2u}{d\theta^2} = \frac{1}{r_0} \left(\frac{\cos \theta}{\cos^2 \theta} + \frac{2 \sin^2 \theta}{\cos^3 \theta}\right) = \frac{1}{r_0} \left(\frac{\cos^2 \theta + 2 \sin^2 \theta}{\cos^3 \theta}\right) = \frac{1}{r_0} \frac{1 + \sin^2 \theta}{\cos^3 \theta}.$$

Also $-u = -\cos^2 \theta / (r_0 \cos^4 \theta)$, so plugging into the differential equation gives

$$\frac{1}{r_0} \frac{1 + \sin^2 \theta}{\cos^3 \theta} = \frac{1}{r_0} \frac{\cos^2 \theta}{\cos^3 \theta} - \frac{m}{L^2} \frac{(r_0 \cos \theta)^2 k(r_0 \cos \theta)^n}{r_0 \cos^4 \theta},$$

$$\frac{1}{r_0} \frac{2}{\cos^3 \theta} = \frac{mk}{L^2} r_0^{2n} (\cos \theta)^{2+n}.$$ 

For this to be true for all values of $\theta$, the $\cos \theta$ terms must match, so $n = -5$. Then $2/r_0 = -mk/(L^2r_0^3)$, or $r_0 = \sqrt{-mk/(2L^2)}$, which implies $k = -2L^2r_0^3/m$. Note that the force must be attractive, since $k < 0$. 
