1. (a) The force is given by $F(x) = -dV/dx = -k(2x - ax^2)e^{-ax}$. This can also be written as $F = -kx(2 - ax)e^{-ax}$, from which we see that the force is zero (i.e., there are equilibrium points) at $x = 0$ and $x = 2/a$. The second derivative of $V$ is $V'' = (2 - 4ax + a^2x^2)e^{-ax}$, which is positive at $x = 0$ (so there is a local minimum there) and negative at $x = 2/a$ (local max). Since the force goes to infinity as $x \to -\infty$ and to zero as $x \to +\infty$, the sketch looks something like this:

![Graph of V(x) plotted for k = 1 and a = 1](image)

(b) In order for the particle to reach $x = +\infty$, it must have enough total energy to get past the local maximum at $x = 2/a$. The potential energy there is $V(2/a) = 4k/(a^2e^2)$. The initial energy is all kinetic (since $V(0) = 0$), so we must have

$$\frac{1}{2}mv_0^2 > \frac{4k}{a^2e^2}.$$ 

This answer does not depend on the initial direction of the particle. If it is initially moving to the left, it will reach a turning point for some negative value of $x$ and return to $x = 0$ with the same speed it had originally, which is then identical to the case where it was initially moving to the right.

2. (a) Typically one guesses a form for the particular solution that matches the nature of the driving force. In this case the driving force is constant, so we guess $x = C$ for the particular solution. Plugging this into the differential equation gives $0 + 0 + kC = -mg$, or $C = -mg/k$. Thus the particular solution is $x = -mg/k$. The general solution is a sum of the homogeneous and particular solutions. Since the homogeneous equation is just a damped harmonic oscillator we have

$$x(t) = (A\cos \omega t + B\sin \omega t)e^{-\gamma t} - \frac{mg}{k},$$

where $\omega = \sqrt{\omega_0^2 - \gamma^2}$, $\omega_0 = \sqrt{k/m}$ and $\gamma = b/(2m)$. By taking a derivative we get

$$v(t) = \frac{dx}{dt} = (-\omega A\sin \omega t + \omega B\cos \omega t - \gamma A\cos \omega t - \gamma B\sin \omega t)e^{-\gamma t},$$
\[ x_0 = A - \frac{mg}{k} \]
\[ v_0 = \omega B - \gamma A, \]

which may be solved to get \( A = x_0 + (mg/k) \) and \( B = (v_0 + \gamma A)/\omega \), so

\[ x(t) = \left\{ (x_0 + \frac{mg}{k}) \cos \omega t + \frac{1}{\omega} \left[ v_0 + \gamma \left( x_0 + \frac{mg}{k} \right) \right] \sin \omega t \right\} e^{-\gamma t} - \frac{mg}{k}. \]

(b) If we define a new coordinate \( x' = x + (mg/k) \), then the solution becomes

\[ x'(t) = \left[ x'_0 \cos \omega t + \frac{1}{\omega} (v_0 + \gamma x'_0) \sin \omega t \right] e^{-\gamma t}, \]

which is a damped harmonic oscillator with its equilibrium point at \( x' = 0 \). The shift is just due to the change in the equilibrium point that occurs when the oscillator is placed in a vertical gravitational field.

3. (a) The differential equation is \( m\ddot{x} + kx = F_0 e^{-at} \). From the form of \( F(t) \), we guess \( x(t) = Ae^{-at} \) for the particular solution. Plugging this into the differential equation gives

\[ ma^2 A e^{-at} + kA e^{-at} = F_0 e^{-at} \]
\[ (ma^2 + k)A e^{-at} = \frac{F_0}{ma^2 + k} \]
\[ A = \frac{F_0}{ma^2 + k}. \]

(b) The homogeneous equation is just a simple harmonic oscillator (undamped). The general solution is the sum of the particular and homogeneous solutions:

\[ x(t) = A \cos \omega_0 t + B \sin \omega_0 t + \frac{F_0}{ma^2 + k} e^{-at} \]
\[ v(t) = \frac{dx}{dt} = \omega_0 (-A \sin \omega_0 t + B \cos \omega_0 t) - \frac{aF_0}{ma^2 + k} e^{-at}, \]

where \( \omega_0 = \sqrt{k/m} \). The initial conditions give

\[ 0 = A + \frac{F_0}{ma^2 + k} \quad \Rightarrow \quad A = -\frac{F_0}{ma^2 + k} \]
\[ 0 = \omega_0 B - \frac{aF_0}{ma^2 + k} \quad \Rightarrow \quad B = \frac{aF_0}{\omega_0 (ma^2 + k)}, \]

so the solution is

\[ x(t) = \frac{F_0}{ma^2 + k} \left[ -\cos \omega_0 t + \frac{a}{\omega_0} \sin \omega_0 t + e^{-at} \right]. \]