1. A particle of mass $m$ is attached to a spring of spring constant $k$, has drag force given by $F(v) = -bv$, and is also subject to a periodic driving force given by

$$F(t) = \begin{cases} F_0 & \text{for } 0 < t \leq T/2 \\ 0 & \text{for } T/2 < t \leq T \end{cases}$$

where $F_0$ is a constant, and which repeats with period $T$ at later times.

(a) Expand the driving force in a Fourier series with angular frequencies given by integer multiples of $2\pi/T$.

(b) Assume that the resonance frequency of the oscillator is given by $\omega_0 = \sqrt{k/m}$, and that $\omega_0 = 6\pi/T$. Use your result in (a) to write down the steady state solution (i.e., ignore the transient term). [Hint: the relationship between Eqs. 2.202 and 2.203 in the text may be helpful!] If the damping is small, show that the motion is approximately sinusoidal with period $T/3$. [Hint: does one term in the Fourier series dominate?]

2. (a) Find the Green’s function for the homogeneous differential equation $m\ddot{x} + b\dot{x} = 0$, describing a particle of mass $m$ with a linear drag force. [Hint: solve the equation assuming the particle is initially at rest and given an impulse $mv_0$ at $t = t'$.]

(b) Use your result from (a) to solve the differential equation $m\ddot{x} + b\dot{x} = F(t)$, where

$$F(t) = \begin{cases} F_0 t/T & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

and $F_0$ and $T$ are constants, assuming $x(0) = 0$ and $v(0) = 0$.

3. (a) Use a Runge-Kutta routine to find $x$ and $v$ at $t = 10$ s for a damped harmonic oscillator with $m = 1$ kg, $k = 1$ N/m for linear damping coefficient $b = 0, 0.1, 1.0,$ and $10$ kg/s, assuming $x(0) = 0$ and $v(0) = 1$ m/s. By how much does your answer deviate from the exact answer for $b = 0.1$? To what can you attribute this error, e.g., is it due to taking too large a step size or is it round-off error? Justify your answer. [Hint: try different step sizes.]

(b) Identify which values of $b$ have underdamping and which have overdamping. For $b = 1$ kg/s determine the period to the nearest 0.01 s, and compare to the exact result.

(c) Now change the damping force to be quadratic in $v$, i.e., $-bv|v|$. All other things being equal (i.e., using the same values for $m, k, x_0$ and $v_0$ as in (a) above), which of the $b$ values above (now in units of kg/m) lead to overdamped motion?