1. A particle of mass $m$ slides without friction down a hill which follows the equation $y = x^{2/3}$. The uniform force of gravity $mg$ acts in the negative $y$ direction.

(a) Using the Lagrange multiplier technique in Cartesian coordinates, derive the equations of motion in $x$ and $y$. What is the constraint equation?

(b) Eliminate $y$ and the Lagrange multiplier from the equations of motion and solve for $\dot{x}$, assuming the particle starts at rest at the point $(x, y) = (x_0, y_0 = x_0^{2/3})$ where $x_0 > 0$.

(c) Use the result in (b) to solve for the Lagrange multiplier, and show that the magnitude of the constraint force is

$$|F| = 3mg \frac{9x^{4/3} + 8x^{2/3} - 4x_0^{2/3}}{x^{1/3} [9x^{2/3} + 4]^{3/2}} = 3mg \frac{9y^2 + 8y - 4y_0}{\sqrt[3]{[9y + 4]^{3/2}}}.$$

Show that the constraint force is always perpendicular to the motion of the particle.

(d) Does the particle ever leave the surface, and if so, where?

2. A particle of mass $m$ is constrained to be in the moving plane described by $z = bx + cy + at^2/2$. The particle is also subject to a constant gravitational force $mg$ acting in the negative $z$ direction. There is no friction.

(a) Using the Lagrange multiplier method with standard Cartesian coordinates, solve for the position of the particle as a function of time, assuming $x_0 = y_0 = z_0 = 0$ and $\dot{x}_0 = \dot{y}_0 = \dot{z}_0 = 0$.

(b) Show that the energy function $h = \sum \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} - L$ is not conserved, i.e., show that $\dot{h} \neq 0$. This can be understood by noting that the forces of constraint do work; calculate the work done per unit time by the constraint forces and show that it is equal to $\dot{h}$.